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On HSS-Based Iteration Methods for Two Classes of Tensor Equations

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Abstract. HSS-based iteration methods for large systems of tensor equations $\mathscr{T}(x) = b$ and $Ax = \mathscr{T}(x) + b$ are considered and conditions of their local convergence are presented. Numerical experiments show that for the equations $\mathscr{T}(x) = b$, the Newton-HSS method outperforms the Newton-GMRES method. For nonlinear convection-diffusion equations the methods based on HSS iterations are generally more efficient and robust than the Newton-GMRES method.

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Key words: Tensor equation, HSS iteration, k-mode product, convergence, large sparse system.

1. Introduction

We consider numerical methods based on HSS iterations for two classes of tensor equations. Let us start with definitions and auxiliary results.

Definition 1.1 (cf. Refs. [14, 19, 23, 24, 29]). We say that $\mathscr{A}$ is a real or complex tensor of order-m dimension-n and write $\mathscr{A} \in \mathbb{R}^{[m,n]}$ or $\mathscr{A} \in \mathbb{C}^{[m,n]}$, if its entries $\mathscr{A}_{i_1,\dots,i_m}$, $i_j = 1,\dots,n$, $j = 1,\dots,m$ belong to the set of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, respectively.

Thus order-0 tensor is a scale, order-1 tensor is a vector and order-2 tensor is a matrix.

Definition 1.2 (cf. Ding & Wei [12]). A tensor $\mathscr{A}$ is said to be diagonal if

$$\mathscr{A}_{i_1,\dots,i_m} = 0 \quad \text{for} \quad \delta_{i_1,\dots,i_m} = 0,$$

where

$$\delta_{i_1,\dots,i_m} = \begin{cases} 1, & \text{if} \quad i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

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In particular, the identity (zero) tensor is the diagonal tensor, all diagonal entries of which are equal to one (zero).

Definition 1.3 (cf. Refs. [4, 18, 26, 34]). The k-mode product of tensor $\mathscr{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_m}$ and vector $x \in \mathbb{R}^{I_k}$ denoted by $\mathscr{A}\bar{\times}_k x$, is the tensor of order-$(m-1)$ with $i_1 \ldots i_{k-1} i_{k+1} \ldots i_m$-components

$$(\mathscr{A}\bar{\times}_k x)_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_m} = \sum_{i_k=1}^{I_k} \mathscr{A}_{i_1 \ldots i_{k-1} i_k i_{k+1} \ldots i_m} x_{i_k},$$

where $k \leq m$ and $I_j, j = 1, \ldots, m$ are positive integers.

In what follows we use the following notations.

Notation 1. If $\mathscr{A} \in \mathbb{R}^{[m,n]}$ and $b \in \mathbb{R}^n$, then

$$\mathscr{A}x^m := \sum_{i_1, i_2, \ldots, i_m = 1}^{n} \mathscr{A}_{i_1, i_2, \ldots, i_m} x_{i_1} x_{i_2} \ldots x_{i_m} \quad \text{is a scale,} \tag{1.1}$$

$$\left(\mathscr{A}x^{m-1}\right)_i := \sum_{i_2, \ldots, i_m = 1}^{n} \mathscr{A}_{i, i_2, \ldots, i_m} x_{i_2} \ldots x_{i_m} \quad \text{is a vector,} \tag{1.2}$$

$$\left(\mathscr{A}x^{m-2}\right)_{i,j} := \sum_{i_3, \ldots, i_m = 1}^{n} \mathscr{A}_{i, j, i_3, \ldots, i_m} x_{i_3} \ldots x_{i_m} \quad \text{is a matrix.}$$

Notations (1.1) and (1.2) are introduced by Qi [29] and have been written as

- $\mathscr{A}x^m := \mathscr{A}\bar{\times}_m x \bar{\times}_{m-1} x \bar{\times}_{m-2} \cdots \bar{\times}_3 x \bar{\times}_2 x \bar{\times}_1 x$ (scale),
- $\mathscr{A}x^{m-1} := \mathscr{A}\bar{\times}_m x \bar{\times}_{m-1} x \bar{\times}_{m-2} \cdots \bar{\times}_3 x \bar{\times}_2 x$ (vector),
- $\mathscr{A}x^{m-2} := \mathscr{A}\bar{\times}_m x \bar{\times}_{m-1} x \bar{\times}_{m-2} \cdots \bar{\times}_3 x$ (matrix)

later on — cf. [11, 25, 27].

Definition 1.4 (cf. Refs. [20, 25, 27]). The equation

$$\mathscr{A}_1 x^{m-1} + \mathscr{A}_2 x^{m-2} + \mathscr{A}_3 x^{m-3} + \cdots + \mathscr{A}_{m-1} x + \mathscr{A}_m = 0, \quad \mathscr{A}_1 \neq 0 \tag{1.3}$$

is called a real (complex) tensor equation of order m if for all $1 \leqslant i \leqslant m$ one has $\mathscr{A}_i \in \mathbb{R}^{[m-i+1,n]}$, $x \in \mathbb{R}^n$ ($\mathscr{A}_i \in \mathbb{C}^{[m-i+1,n]}, x \in \mathbb{C}^n$), where

$$\mathscr{A}_i x^{m-i} = \mathscr{A}_i \bar{\times}_{m-i+1} x \bar{\times}_{m-i} x \bar{\times}_{m-i-1} \cdots \bar{\times}_3 x \bar{\times}_2 x, \quad 1 \leqslant i \leqslant m. \tag{1.4}$$

Note that tensor notations can be used to represent Taylor polynomials of multivariable functions. Thus if Ω is a convex set and $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a k-time differentiable function, then we can write the Taylor's expansion of F around $x = x_c \in \mathbb{R}^n$ as

$$\begin{aligned} F(x) &= \sum_{i=0}^{k} \frac{1}{i!} F^{(i)}(x_c)(x - x_c)^i + o(\|x - x_c\|^k) \\ &= F(x_c) + F'(x_c)(x - x_c) + \cdots + \frac{1}{k!} F^{(k)}(x_c)(x - x_c)^k + o(\|x - x_c\|^k), \end{aligned}$$