East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.090419.041219

Vol. **10**, No. 3, pp. 455-484
August 2020

A Hybridisable Discontinuous Galerkin Method for Time-Dependent Convection-Diffusion-Reaction Equations

Wei Pi, Yihui Han and Shiquan Zhang[1,*]

School of Mathematics, Sichuan University, Chengdu 610064, China.

Received 9 April 2019; Accepted (in revised version) 4 December 2019.

Abstract. A hybridisable discontinuous Galerkin (HDG) discretisation of time-dependent linear convection-diffusion-reaction equations is considered. For the space discretisation, the HDG method uses piecewise polynomials of degrees $k \geq 0$ to approximate potential u and its trace on the inter-element boundaries, and the flux is approximated by piecewise polynomials of degree $\max\{k-1,0\}, k \geq 0$. In the fully discrete scheme, the time derivative is approximated by the backward Euler difference. Error estimates obtained for semi-discrete and fully discrete schemes show that the HDG method converges uniformly with respect to the equation coefficients. Numerical examples confirm the theoretical results.

AMS subject classifications: 65M10, 78A48

Key words: Convection-diffusion-reaction equation, hybridisable discontinuous Galerkin method, semi-discrete, fully discrete, error estimate.

1. Introduction

In this work, we consider a new hybridisable discontinuous Galerkin (HDG) finite element method for the time-dependent linear convection-diffusion-reaction equations

$$
\begin{aligned}
&\partial_t u - \epsilon \Delta u + \beta \cdot \nabla u + cu = f && \text{in} \quad \Omega \times [0,T], \\
&u = g && \text{on} \quad \partial\Omega \times [0,T], \\
&u(\cdot,0) = u_0 && \text{in} \quad \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^d$, $d = 2,3$ is a Lipschitz polyhedral domain with the diameter L, $\beta(x,t) \in C([0,T];W^{1,\infty}(\Omega)^d)$ and $c(x,t) \in C([0,T];L^\infty(\Omega))$ denote the convection and reaction fields, respectively. Besides, ϵ is a positive constant and T the length of the time interval. The functions f, g and u_0 are assumed to be sufficiently smooth. In order to study the stability and convergence od the method, we also need the following conditions — cf. [2,9]:

*Corresponding author. *Email addresses:* 512442593@qq.com (W. Pi), 1335751459@qq.com (Y. Han), shiquanzhang@scu.edu.cn (S. Zhang)

A1. There is a constant σ_0 such that

$$0 < \sigma_0 \leq \sigma := c(x,t) - \frac{1}{2}\nabla \cdot \beta(x,t) \quad \text{for all} \quad x \in \Omega, \quad t \in [0,T].$$

A2. There are positive constants c_β and c_σ such that

$$c_\beta \|\beta\|_{1,\infty} \leq \|\beta\|_{0,\infty}, \quad c_\sigma \|\sigma\|_{0,\infty} \leq \sigma_0 + \beta_0 \quad \text{with} \quad \beta_0 := \frac{\|\beta\|_{0,\infty}}{L}.$$

For steady-state convection-diffusion-reaction problems, the numerical accuracy of standard finite element methods usually deteriorates in convection-dominated regimes. The approaches developed to overcome this issue include stabilised technique [4, 5, 19, 22, 27] and least squares technique [10]. Other types of finite element methods such as DG methods [2, 7, 11, 12, 14, 24, 38], HDG methods [13, 20, 33, 35] and WG method [9] have been also used. All these approaches work well not only for convection-dominated regimes but also for diffusion- dominated, reaction-dominated and intermediate regimes. The streamline upwind Petrov-Galerkin (SUPG) method with an added consistent diffusion term is a popular method for solving time-dependent linear convection-diffusion equations of convection dominated problems [6]. The efficiency of finite element methods for convection-diffusion-reaction equations such as space-time Galerkin/least-squares method [6, 27], the subgrid scale method [21, 25, 26, 39], characteristic Galerkin method [18, 30, 34] and Taylor-Galerkin method [17] has been discussed in [16]. The stability and convergence of the SUPG method for the transient convection-diffusion(-reaction) problems is studied in [3, 28]. Numerical performance of several stabilised finite element methods for scalar time-dependent convection-diffusion-reaction equations with small diffusion is considered in [29]. Similar stabilisation technique is also used when solutions have a singularity — e.g. in time-dependent Maxwell's equations [1], in conservative perturbed MHD model [31] and in modified phase field crystal (MPFC) equation [23]. A hybridisable discontinuous Galerkin (HDG) method was also employed in steady and unsteady convection-diffusion equations and numerical experiments show that the equal-order HDG method is robust with respect to the coefficients ϵ, β [32].

In this work, we investigate a new HDG method for the time-dependent convection-diffusion-reaction problem (1.1) but unlike the approach [32], our method has the following features:

- Instead of an equal order approximations of all variables, we approximate the scalar function u and its trace on the inter-element boundaries by piecewise polynomials of degree $k \geq 0$, and we use piecewise polynomial of degree $\max\{k-1, 0\}$, $k \geq 0$ for the flux approximation.

- The error estimates of semi-discrete and fully discrete HDG schemes show that the method proposed is robust with respect to the coefficients ϵ, β and c. In particular, in the L^2-norm the method has $(k+1/2)$-order of convergence in the convection-dominated case and $k+1$-order in other situations.