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A Novel Spectral Method for Burgers Equation on The Real Line

Yujian Jiao[1,2,*]

[1]*Department of Mathematics, Shanghai Normal University, Shanghai 200234, P.R. China.*
[2]*Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, P.R. China.*

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Abstract. A spectral method for the Burgers equation on the whole real line based on generalised Hermite functions is proposed. The generalised stability and convergence of the method are proved. Numerical results confirm the theoretical findings and demonstrate the efficiency of the algorithm.

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Key words: Burger equation on the real line, spectral method, nonlinear problem, generalised Hermite function.

1. Introduction

The main object of our interest is the equation, which originated in the work of Bateman [6] in 1915 and was lately named after Burgers, who employed it to study turbulent fluids [8]. Nowadays this equation is widely used in various areas of applied mathematics, including gas dynamics, waves in shallow water and turbulence in fluid dynamics [2, 7, 21, 32–34, 36, 37, 39, 40].

A vast literature is devoted to analytic solutions of the Burgers equation on bounded and unbounded domains with different initial and boundary conditions — cf. Refs. [1, 5, 9, 15, 22, 35, 41, 44]. Numerical methods such as finite difference methods [4, 10, 26, 29] and finite element approaches [3, 11, 12, 14, 23, 27, 45] have been also extensively studied. Another popular group of numerical methods for the Burgers equation — viz. spectral methods, attracted a considerable attention as well. In particular, Maday and Quarteroni [30, 31] considered Legendre and Chebyshev spectral and pseudospectral methods, Ma and Guo [28] studied Chebyshev spectral method for Burgers-like equation, Wu *et al.* [42] presented a Chebyshev-Legendre collocation method for generalised Burgers equation and

*Corresponding author. *Email address:* `yj-jiao@shnu.edu.cn` (Y. Jiao)

Khater *et al.* [25] discussed Chebyshev spectral collocation methods for Burgers-type equations. Now the challenge is to develop efficient algorithms for the Burgers equation on unbounded domains and to study their stability and convergence. One approach is related to employing finite difference/finite element methods for equation with artificial boundary conditions [13] and another is domain decomposition methods [19]. However, these approaches may bring additional errors and complicate theoretical analysis and actual computations. Thus the use of certain orthogonal systems on unbounded domains is more suitable. Guo and Xu [20] employed Hermite polynomials and Guo *et al.* [18] developed spectral and pseudospectral methods based on Hermite functions for problems on the real line.

In this work, we engage generalised Hermite functions to approximate the Burgers equation on the real line. The benefit of this approach is three folds. The first is the direct approximation of the solutions on the whole line, so that there is no need to employ domain decomposition or variable transformations. Second, this simplifies theoretical analysis and produces sparse systems. Finally, numerical solutions have the spectral accuracy in space.

The remainder of the paper is organised as follows. In Section 2, we recall approximation results for generalised Hermite functions. A spectral scheme for the Burgers equation is introduced and analysed in Section 3. Numerical results presented in Section 4, demonstrate the efficiency of this algorithm, whilst Section 5 contains some concluding remarks.

2. Preliminaries

Let $\Lambda = \{\xi | -\infty < \xi < \infty\}$ and $\chi(\xi)$ be a weight function. For an integer $\mu \geq 0$, we denote by $(u,v)_{\mu,\chi,\Lambda}$, $|u|_{\mu,\chi,\Lambda}$ and $\|u\|_{\mu,\chi,\Lambda}$ the inner product, semi-norm and norm of the space $H^{\mu}_{\chi}(\Lambda)$, respectively. In particular, $H^0_{\chi}(\Lambda) = L^2_{\chi}(\Lambda)$ has inner product $(u,v)_{\chi,\Lambda}$ and the norm $\|u\|_{\chi,\Lambda}$. For simplicity, we omit subscript χ whenever $\chi(\xi) \equiv 1$. The l-th order Hermite polynomial is given by

$$\hbar_l(\xi) = (-1)^n e^{\xi^2} \partial^l_{\xi} \left(e^{-\xi^2} \right).$$

Such polynomials are mutually orthogonal with respect to the weight $\chi(\xi) = e^{-\xi^2}$. Following [43], we now consider the generalised Hermite functions

$$\hbar^{\sigma}_l(\xi) = \frac{1}{\sqrt{2^l l!}} \hbar_l(\sigma\xi) e^{-(1/2)\sigma^2\xi^2}, \quad l \geq 0, \tag{2.1}$$

where $\sigma > 0$ is a constant. According to [38, Eq. (7.65)] and the definition (2.1), we have

$$\hbar^{\sigma}_l(\xi) = \sqrt{2^l l!} \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^{2k} k! (l-2k)!} \left((\sigma\xi)^{l-2k} e^{-(\sigma^2\xi^2)/2} \right).$$

Obviously,

$$\lim_{|\xi|\to\infty} \hbar^{\sigma}_l(\xi) = 0, \quad l \geq 0. \tag{2.2}$$