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An Efficient Newton Multiscale Multigrid Method for 2D Semilinear Poisson Equations

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Abstract. An efficient Newton multiscale multigrid (Newton-MSMG) for solving large nonlinear systems arising in the fourth-order compact difference discretisation of 2D semilinear Poisson equations is presented. The Newton-MG method is employed to calculate approximation solutions on coarse and fine grids and then a completed Richardson extrapolation is used to construct a sixth-order extrapolated solution on the entire fine grid directly. The method is applied to two nonlinear Poisson-Boltzmann equations and numerical simulations show that the Newton-MSMG method is a cost-effective approach with the sixth-order accuracy.

AMS subject classifications: 65N30, 65N55

Key words: Semilinear Poisson equation, Richardson extrapolation, sixth-order accuracy, Newton's method, multiscale multigrid.

1. Introduction

Numerical solution of Poisson equations plays an important role in electrostatics and mechanical engineering. In particular, the multigrid (MG) methods combined with Richardson extrapolation strategies are fast and provide high-accuracy solutions of Poisson equations. Thus Chen *et al.* [2] and Hu *et al.* [9,10] considered an extrapolation cascadic multigrid (EXCMG) algorithm based on finite element (FE) error expansions. In the EXCMG method, an extrapolation operator for the conjugate gradient (CG) method is developed to construct an accurate initial guess — i.e. a high-order approximation of an FE solution on the next finer grid. This greatly accelerate the convergence of the original cascadic multigrid algorithm. Later on, Pan *et al.* [14,15] and Li *et al.* [11,12] extended the EXCMG al-

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gorithm to high-order compact difference schemes for 2D and 3D Poisson equations. Wang and Zhang [23] proposed a multiscale multigrid (MSMG) method for 2D Poisson equations. The MSMG method employs a multigrid V-cycle procedure to compute fourth-order accurate solutions on the fine and coarse grids. Using an iterative refinement procedure with the Richardson extrapolation technique, it generates a sixth-order accurate solution on the fine grid. Numerical simulations show the efficacy and high accuracy of the MSMG method for finding solutions of Poisson and convection-diffusion equations [3, 5, 6, 24]. Recently, Dai *et al.* [4] applied the EXCMG method to determine an initial guess for the original MSMG method. They also introduced an EXCMG accelerated multiscale multigrid (EXCMG-MSMG) method, which is more efficient than the MSMG method.

Although such extrapolation multigrid methods can simultaneously achieve high accuracy and high efficiency, they are only applicable to linear problems. On the other hand, the existing solvers for nonlinear problems, such as Newton multigrid (Newton-MG) method [1], cascadic multigrid (CMG) methods [18, 21, 26, 29] and adaptive multigrid methods [8, 25], cannot achieve the sixth-order accuracy. Therefore, it is important to generalise the extrapolation multigrid algorithms to nonlinear problems.

In this work, we employ fourth-order compact difference schemes and develop a Newton multiscale multigrid (Newton-MSMG) method, which achieves the sixth-order accuracy in solving 2D semilinear Poisson equations. Starting with the Newton-MG method, we determine fourth-order accurate solutions on coarse and fine grids, and then use the completed Richardson extrapolation to generate sixth-order accurate solutions on the entire fine grid directly.

The outline of our presentation is as follows. Section 2 introduces fourth-order compact difference schemes for 2D semilinear Poisson equations. A completed Richardson extrapolation operator is considered in Section 3. A Newton-MSMG method for the semilinear Poisson equation is described in Section 4. Numerical results in Section 5 show the efficiency and the accuracy of the method. Some conclusions are given in the final section.

2. Fourth-Order Compact Difference Schemes

We consider the following 2D semilinear Poisson equation

$$u_{xx}(x,y)+u_{yy}(x,y)=g(u,x,y), \quad (x,y)\in\Omega, \tag{2.1}$$

where the nonlinear forcing function $g(u,x,y)$ and unknown function $u(x,y)$ are assumed to be continuously differentiable and have required partial derivatives. We impose the Dirichlet boundary condition on $\partial\Omega$.

For simplicity, we assume that Ω is a rectangular domain $[L_a,L_b]\times[L_c,L_d]$ subdivided into uniform grid Ω_h with mesh sizes

$$h_x=\frac{1}{N_x}(L_b-L_a), \quad h_y=\frac{1}{N_y}(L_d-L_c),$$

where N_x and N_y are the numbers of uniform intervals in the x and y directions, respectively. Let $U_{i,j}$ denote an approximation of u at the mesh point (x_i,y_j) with $x_i=L_a+ih_x$