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Riemann-Hilbert Approach and N-Soliton Solutions For Three-Component Coupled Hirota Equations

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Abstract. A Riemann-Hilbert problem is employed to study integrable three-component coupled Hirota (tcCH) equations. Thus, we investigate the spectral properties of tcCH equations with a 4×4 Lax pair and derive a Riemann-Hilbert problem, the solution of which is used in constructing N-soliton solutions of the tcCH equations. While considering the spatiotemporal evolution of scattering data, the symmetry of the spectral problem is exploited. Graphical examples show new phenomena in soliton collision, including localised structures and dynamic behaviors of one- and two-soliton solutions. The results can be of interest in nonlinear dynamics of N-component nonlinear Schrödinger type equations.

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Key words: Three-component coupled Hirota equation, Riemann-Hilbert approach, N-soliton solution.

1. Introduction

The Hirota equation [8] can be written in dimensionless form as

$$iq_t + \frac{1}{2}q_{tt} + |q|^2 q - i\alpha_3 q_{ttt} - i6\alpha_3|q|^2 q_t = 0, \tag{1.1}$$

cf. [1]. If $\alpha_3 = 0$, it reduces to the nonlinear Schrödinger equation, which describes a plane self-focusing and one-dimensional self-modulation of waves in hydrodynamics and optical fiber transmission. The coupled Hirota equations

$$\begin{aligned} &iu_t + \frac{1}{2}u_{xx} + \left(|u|^2 + |v|^2\right)u + i\epsilon\left[u_{xxx} + \left(6|u|^2 + 3|v|^2\right)u_x + 3uv^* v_x\right] = 0, \\ &iv_t + \frac{1}{2}v_{xx} + \left(|u|^2 + |v|^2\right)v + i\epsilon\left[v_{xxx} + \left(6|u|^2 + 3|v|^2\right)v_x + 3vu^* u_x\right] = 0 \end{aligned} \tag{1.2}$$

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describe the pulse propagation in a coupled fiber with higher-order dispersion and self-steepening. Various properties of such equations have been studied by the Darboux transformation [15]. However, since the coupled Hirota equations cannot be used for solving compatibility problems of the wavelength division multiplexing linear systems [2], the following three-component coupled Hirota equations have been introduced:

$$\begin{aligned}
& iq_{1t}+\frac{1}{2}q_{1xx}+\left(|q_1|^2+|q_2|^2+|q_3|^2\right)|q_1| \\
&\qquad +i\epsilon\Big[q_{1xxx}+3\left(2|q_1|^2+|q_2|^2+|q_3|^2\right)q_{1x}+3q_1\left(q_2^*q_{2x}+q_3^*q_{3x}\right)\Big]=0, \\
& iq_{2t}+\frac{1}{2}q_{2xx}+\left(|q_1|^2+|q_2|^2+|q_3|^2\right)|q_2| \\
&\qquad +i\epsilon\Big[q_{2xxx}+3\left(2|q_2|^2+|q_3|^2+|q_1|^2\right)q_{2x}+3q_2\left(q_3^*q_{3x}+q_1^*q_{1x}\right)\Big]=0, \\
& iq_{3t}+\frac{1}{2}q_{3xx}+\left(|q_1|^2+|q_2|^2+|q_3|^2\right)|q_3| \\
&\qquad +i\epsilon\Big[q_{3xxx}+3\left(2|q_3|^2+|q_1|^2+|q_2|^2\right)q_{3x}+3q_3\left(q_1^*q_{1x}+q_2^*q_{2x}\right)\Big]=0,
\end{aligned} \tag{1.3}$$

where $q_1(x)$, $q_2(x)$ and $q_3(x)$ are complex envelops, q_i^*, $i=1,2,3$ the complex conjugates of q_i, and the small dimensionless real parameter ϵ denotes the strength of high-order effects. The Lax pair of the tcCH equations has been derived by Bindu [2] and breather wave solutions of the tcCH equations (1.3) by Xu and Chen [26].

The inverse scattering transformation is a powerful analytical tool to solve integrable systems in nonlinear sciences. The Riemann-Hilbert approach developed by Zakharov *et al.* [32] is based on inverse scattering transformation of integrable systems. Along with other methods, it was used for solving various integrable models [4–7, 9–14, 16–25, 27–30, 33–39]. In this work we use the Riemann-Hilbert approach to determine new abundant N-soliton solutions of the tcCH equations (1.3) and display their propagation behavior.

This work is structured as follows. In Section 2, we analyse the spectrum problem of the tcCH equations and establish analytical properties of the Jost functions. These properties are used to rewrite the equations under consideration as a Riemann-Hilbert problem. Moreover, the symmetry of the scattering matrix and time-spatial revolutions of the scattering data are studied. In Section 4, we derive solutions of the corresponding Riemann-Hilbert problem thus obtaining N-soliton solution of the tcCH equations (1.3). Besides, the propagation of soliton solutions is discussed for single- and two-soliton solutions. Some conclusions and discussions are presented in the final section.

2. Direct Scattering Transform

Here we study the Riemann-Hilbert problem for the Eq. (1.3) by using the direct scattering transform. The Lax pair of the tcCH equations is

$$\begin{aligned}
&\Phi_x=U\Phi, \quad U=\lambda U_0+U_1, \\
&\Phi_t=V\Phi, \quad V=\lambda^3V_0+\lambda^2V_1+\lambda V_2+V_3,
\end{aligned} \tag{2.1}$$