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Artificial Boundary Method for European Pricing Option Problem

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Abstract. This paper deals with the valuation of European call options under the Heston stochastic volatility model. An asymptotic solution of the European pricing option problem in powers of the volatility of variance is derived. An artificial boundary method for solving the problem on a truncated domain is considered and artificial boundary conditions are constructed. Numerical simulations show that these conditions allow to find more accurate numerical solutions than for the widely-used Heston boundary condition.

AMS subject classifications: 65M06, 35C20, 35K20

Key words: Option pricing, Heston model, asymptotic analysis, artificial boundary condition.

1. Introduction

Derivatives play an important role in financial markets and their valuation attracted a considerable attention. The derivative is a financial product whose value depends on the price of various underlying assets. One example of derivative is the European call option which gives the holder the right to buy the underlying asset on the expiration date at a price fixed in advance. Option valuation was mainly based on intuition before Black and Scholes [5] and Merton [41] made a major breakthrough in 1973. In particular, they assumed that the asset price follows a geometric Brownian motion and used Itô's theory to derive an initial-boundary problem for partial differential equation describing the European option. The solution of the problem, known as the Black-Scholes formula, has been widely used in industry since then.

However, due to strong assumptions on constant drift and volatility, the Black-Scholes-Merton (BSM) model fails to capture the skewness and kurtosis of asset prices and the volatility smile observed in real markets. Numerous further studies led to two main types of models with improved features — viz. local volatility and stochastic volatility models. In particular, the local volatility model proposed by Dupire [12] and Derman and Kani [10]

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considers the volatility as a deterministic function that depends on both the stock price and time. Although such a model can be easily calibrated according to the market data, it is not capable to capture the dynamics of the smile — cf. [22]. Therefore, stochastic volatility models are usually preferred in practice and the list of stochastic processes, adopted to describe the volatility, includes Ornstein-Uhlenbeck (OU) process in the Stein-Stein model [46] and the Cox-Ingersoll-Ross (CIR) process in the Heston model [33]. Both of them demonstrate a good performance on reflecting the real markets and have closed-form solutions for the European option. One advantage of the Heston model is that the CIR process guaranties the nonnegativity of the volatility, while in OU processes and in the Stein-Stein model the volatility can go negative. Consequently, Heston model gained considerable attention and become extremely popular in research and practice.

The closed-form solution of the option pricing problem for European options under Heston model presented in [33] involves two multivalued functions — viz. the square root and complex logarithm. The functions can have discontinuity points when principal values are considered, so they require a careful treatment. The problem was first mentioned in [42] and further discussed in [1,19,38]. As the result, asymptotic analysis and numerical methods were invoked in order to get a better understanding of the option price. Fouque *et al.* [17] developed a framework for studying the asymptotic behavior of stochastic volatility models. Han *et al.* [31] used this approach and introduced an asymptotic expansion of the option price for a fast mean-reverting process in the Heston model. Zhang *et al.* [50] pointed out that for a slow mean-reverting processes the asymptotic solution has the same form as in [31]. On the other hand, for options with early exercise features and some exotic options it is difficult to establish analytic closed-form and asymptotic solutions. Therefore, there are various numerical methods to price options under the Heston model, including lattice approach [16,40,47], Monte Carlo simulation methods [2,7,45] and the FFT-based methods [8,15]. Since the Heston model leads to a two-dimensional convection-diffusion equation with a mixed second-order derivative, finite difference and finite element methods such as a Crank-Nicolson scheme [54], alternating direction implicit schemes [34], the method of lines [9], a mixture of standard Galerkin finite element and finite volume methods [53], a spectral element method [52] have been adopted. It is worth noting that finite difference methods are more interpretable, flexible, require less pre-computations and can be easily implemented.

However, the corresponding initial boundary value problem is defined on an infinite domain and has to be handled carefully. There are many numerical methods for partial differential equations on unbounded domains, including domain truncation [3, 14], coordinate transforms [21], infinite element methods [4] and spectral methods [6, 43]. One of the most efficient is the domain truncation and introduction of artificial boundary conditions. The method was introduced by Engquist and Majda [14] in 1977 when considering the wave equation. Since then, it has been applied to the Laplace equation [25,29], Helmholtz equation [20,49], Schrödinger equation [3,24], Klein-Gordon equation [28,30] and so on. For more information about the method we refer the reader to [27]. As far as the option valuation problem is concerned, Han and Wu [26] proposed a method to provide artificial boundary conditions for BSM model. Wong and Zhao [48] extended Han and Wu approach