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Convergence of Finite Difference Method in Positive Time for Multi-Term Time Fractional Differential Equations

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Abstract. A multi-time fractional-order reaction-diffusion equation with the Caputo fractional derivative is considered. On a uniform grid, the problem is discretised by using the $L1$ formula. For the problem solutions with a singularity at time $t=0$, the convergence order is $\mathcal{O}(\tau^{\alpha_1})$. For any subdomain bounded away from $t=0$, the method has the convergence rate $\mathcal{O}(\tau)$, which is better than the convergence rate $\mathcal{O}(\tau^{\alpha_1})$ for the whole time-space domain. Results of theoretical analysis are illustrated by numerical experiments.

AMS subject classifications: 65M06, 65M12, 65M15, 26A33

Key words: Caputo fractional derivative, multi-term fractional differential equation, weak singularity, uniform mesh, L1 scheme.

1. Introduction

We consider the following multi-term time fractional differential equations with the Caputo fractional derivative

$$\begin{aligned} &\sum_{i=1}^{l}\left[q_i D_t^{\alpha_i}u\right]-p(x)\frac{\partial^2 u}{\partial x^2}+r(x)u=f(x,t), && (x,t)\in(0,L)\times(0,T],\\ &u(0,t)=u(L,t)=0, && t\in(0,T],\\ &u(x,0)=u_0(x), && x\in(0,L), \end{aligned} \tag{1.1}$$

where

$$D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\frac{\partial u}{\partial s}ds,$$

$0<\alpha_i<1$ and q_i are positive integers [8, 13]. Without loss of generality, we assume that $q_1=1$, $0<\alpha_l<\cdots<\alpha_2<\alpha_1<1$ and $p(x)\geq p_0>0, r(x)\geq r_0>0$ are smooth functions.

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In the special case $l = 1$, the singularity of the solution was determined in [15]. Moreover, for a graded mesh it was shown that the convergence order is $\mathscr{O}(N^{-\min\{\gamma\alpha,2-\alpha\}})$, where N, γ and $\alpha \in (0,1)$ represent the number of time splits, the graded grid parameter, and the order of time fractional derivative, respectively. In [1], the time fractional derivative is discretised by the $L2-1_\sigma$ formula on graded meshes with the convergence order $\mathscr{O}(N^{-\min\{\gamma\alpha,2\}})$. For smooth and non-smooth initial data, Jin *et al.* [10] establish the convergence rate $\mathscr{O}(\tau)$ in the L^2 norm. Yan *et al.* [16] employed a modified $L1$ formula to time-fractional partial differential equations and showed that the method converges as $\mathscr{O}(\tau^{2-\alpha})$. Jin *et al.* [9] considered two discrete schemes using the spatial piecewise linear Galerkin finite element method and time convolutional quadratures. Gracia *et al.* [6,7] proved that for any sub-domain bounded away from $t = 0$, the convergence rate is higher than the one for the entire time-space domain. On the other hand, for $l \neq 1$ the existence and uniqueness of solutions are considered in [12,14]. Note that Huang *et al.* [8] showed the singularity of the solution at the point $t = 0$ and established the convergence order $\mathscr{O}(N^{-\min\{\gamma\alpha_1,2-\alpha_1\}})$ for discretisation on the graded mesh, Cui [2] studied the accuracy of finite difference schemes on graded and uniform meshes for single and multi-term time-fractional diffusion equations with variable coefficients and non-smooth solutions, Gao [4] proposed a compact difference scheme for fourth-order temporal multi-term fractional wave equations and Gao and Yang [5] introduced a fast evaluation method for linear combinations of Caputo fractional derivatives and applied it to multi-term time-fractional sub-diffusion equations.

However, to the best of the authors' knowledge, there are only a few works devoted to multi-term time fractional equations. Following the ideas of [7], we consider equations with multi-term time fractional derivatives and show that for any sub-domain bounded away from $t = 0$, the maximum node error is about $\mathscr{O}(\tau)$. This is better that the corresponding errors for the whole time domain.

The rest of this work is arranged as follows. In Section 2, we discretise the problem on a uniform grid and describe the discrete scheme. Section 3 deals with the error of the scheme on subintervals. In Section 4, we present the results of numerical examples in order to verify theoretical analysis. Finally, we give conclusion for the article.

2. Discrete Problem

In this section, we discretise the problem (1.1) on a uniform grid and describe the numerical method used. For positive integers M and N, let $h := L/M$, $\tau := T/N$, and

$$\Omega^{M,N} := \big\{(x_m, t_n) | x_m = mh, t_n = n\tau, m = 0,1,2,\ldots,M, n = 0,1,2,\ldots,N\big\}.$$

By u_m^n we denote the value of the solution $u(x,t)$ of the problem (1.1) at the point (x_m, t_n).

Considering the Caputo fractional derivative

$$D_t^\alpha u(x_m,t_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \frac{\partial u(x_m,s)}{\partial s} ds$$