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A Non-Consistent Virtual Element Method for Reaction Diffusion Equations

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Abstract. A simple virtual element method, avoiding the traditional enhancement technique, is used for numerical solution of a reaction-diffusion problem in the lowest order cases $k = 1$ and 2. Optimal error estimates are established in H^1 and L^2 norms. Numerical results are consistent with theoretical findings but show that for $k \geq 3$ the method is not optimal.

AMS subject classifications: 65N30, 65N15

Key words: Virtual element method, reaction-diffusion problem, non-consistent, enhancement technique, error analysis.

1. Introduction

This paper focuses on conforming virtual element methods (VEMs) for the reaction-diffusion problem

$$\begin{aligned} -\Delta u + \alpha u &= f \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where α is a nonnegative constant and Ω is a polygonal domain. Developing mimetic finite difference methods, Beirão Da Veiga *et al.* [3] introduced a virtual element method for the problem (1.1) with $\alpha = 0$. This approach has been further developed in [1, 4]. Such a generalisation of standard finite element methods allows using general polytopal meshes and has other advantages such as better handling of partial differential equations (PDEs) on complex geometric domains and the ones associated with high-regularity admissible spaces. In recent years, conforming and nonconforming VEMs have been used to solve second order elliptic equations [1, 3, 12, 16], fourth order elliptic equations [2, 11, 15, 19], or even $2m$-th order elliptic problems [14].

Let K be a polygon in $\mathbb{R}^2$ with the boundary ∂K and $\mathbb{P}_m(K)$ denote the set of all polynomials on K with the total degree at most m. For convenience, we also set $\mathbb{P}_{-1}(K) := \{0\}$.

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Consider the first virtual element space

$$V_k(K) := \left\{ v \in H^1(K) : \Delta v \in \mathbb{P}_{k-2}(K) \text{ in } K,\ v|_{\partial K} \in \mathbb{B}_k(\partial K) \right\}, \tag{1.2}$$

where

$$\mathbb{B}_k(\partial K) := \left\{ v \in C(\partial K) : v|_e \in \mathbb{P}_k(e),\ e \subset \partial K \right\},$$

and $e \subset \partial K$ is a boundary edge — cf. [3]. Equipped with suitable degrees of freedom (DOFs), a computable approximate bilinear form can be constructed by using the elliptic projector Π_k^∇. To ensure the accuracy and the well-posedness of the discrete method, the k-consistency (2.3) and the stability (2.4) have to be taken into account. In view of the first condition, a natural candidate to approximate the reaction term $(u, v)_K$ in the local bilinear form is $(\Pi_k^0 u, \Pi_k^0 v)_K$, where Π_k^0 is the L^2 projection onto $\mathbb{P}_k(K)$ and $(\cdot,\cdot)_K$ the usual $L^2(K)$ inner product. Nevertheless, Π_k^0 can not be computed in terms of DOFs attached to $V_k(K)$. Therefore, Ahmad *et al.* [1] modified the VEM space (1.2) to a local enhancement space — viz.

$$W_k(K) = \left\{ w \in V_{k,k}(K) : (w - \Pi_k^\nabla w, q)_K = 0,\ q \in \mathbb{M}_k(K) \backslash \mathbb{M}_{k-2}(K) \right\}, \tag{1.3}$$

where, for a nonnegative integer r, $\mathbb{M}_r(K)$ is given by the definition (2) in [1], and

$$V_{k,k}(K) := \left\{ v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K),\ \Delta v \in \mathbb{P}_k(K) \text{ in } K \right\}.$$

In this space, the operator Π_k^0 can be easily computed using Π_k^∇ and the local DOFs related to $W_k(K)$. The key idea is to substitute the redundant DOFs of $\Pi_k^0 v$ by DOFs associated with $\Pi_k^\nabla v$. Such a procedure is now dubbed as the VEM enhancement [12] and is widely used in other problems, including conforming and nonconforming VEMs for convection-reaction-diffusion problems with variable coefficients, implementation of the VEM for general elliptic equations in primal and mixed form, the Morley-type virtual elements for plate bending problem [5, 12, 19], and $2m$-th order elliptic problems [14].

In order to simplify the presentation, we restrict ourselves to two-dimensional problems and a family of polygonal meshes $\{\mathcal{T}_h\}_h$ that satisfies the following condition — cf. [10, 13]:

A1. For each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K such that $\mathcal{T}_K$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_K$ is proportional to h_K. Each edge of K is a side of a certain triangle in $\mathcal{T}_K$.

As shown in [13], this condition comprises usual geometric assumptions frequently used in the context of VEMs.

We are concerned with virtual element methods in the most used cases $k = 1$ and $k = 2$. In contrast to the enhancement technique [12], the elliptic projection $\Pi_k^\nabla v$ is directly used to approximate the reaction term instead of the L^2 projection $\Pi_k^0 v$. The scheme is referred to as the non-consistent method since the k-consistency condition is violated at this time. The error analysis is based on the observation that $V_k(K)$ and $W_k(K)$ share the same degrees of freedom, and $\Pi_k^\nabla v = \Pi_k^0 v$ for all $v \in W_k(K)$ when $k = 1, 2$. This implies that the solutions of this non-consistent method and an enhancement virtual element method take the same values for the degrees of freedom. Afterwards, using error estimates for the latter method