East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.030120.120520

Vol. **10**, No. 4, pp. 800-817
November 2020

Two-Grid Finite Element Methods of Crank-Nicolson Galerkin Approximation for a Nonlinear Parabolic Equation

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Received 3 January 2020; Accepted (in revised version) 12 May 2020.

Abstract. Two-grid finite element methods with the Crank-Nicolson Galerkin scheme for nonlinear parabolic equations are studied. It is shown that the methods have convergence order $\mathscr{O}(h + H^2 + (\Delta t)^2)$ in H^1-norm, so that a larger time step can be used in numerical calculations. In addition to saving computing time, the algorithms provide a good approximation of the problem solution and numerical examples confirm their efficiency.

AMS subject classifications: 65N15, 65N30

Key words: Nonlinear parabolic equation, finite element method, two-grid, Crank-Nicolson Galerkin scheme, optimal convergence order.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with the smooth boundary $\partial\Omega$. We consider the initial-boundary value problem for the following nonlinear parabolic equation

$$\begin{aligned} & u_t - \nabla\cdot(a(u)\nabla u) = f(u), && (x,t) \in \Omega \times (0,T], \\ & u(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T], \\ & u(x,0) = u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where $u_t = \partial u/\partial t$ and f is a given real-valued function. We assume that $a(u)$ is sufficiently smooth and bounded — i.e. there exist positive constants a_0 and a_1 such that

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$$0 < a_0 \le a(u) \le a_1.$$

In addition, we also assume that the functions $a(u)$, $f(u)$,

$$a_u(u) := \frac{\partial}{\partial u} a(u), \quad a_{uu}(u) := \frac{\partial^2}{\partial u^2} a(u)$$

satisfy (globally) the Lipschitz condition and the problem (1.1) has a sufficiently smooth unique solution [25].

Nonlinear parabolic equations (NPE) arise in various applied fields. In particular, Keller and Segel [15] used such equations in a chemotaxis model and Chavent and Jaffré [2] in simulation of oil recovery techniques. Numerical methods for nonlinear parabolic equations also attracted considerable attention. Thus Luskin [22] studied a Galerkin method for equations with nonlinear boundary conditions, Hayes [14] analysed a modified backward time discretisation using patch approximations, Li [18] considered a fully discrete finite element method and established L^2 and H^1 error estimates, Farago [13] determined properties of semidiscrete approximations in a finite element method, Liu [21] obtained L^2 and H^1 error estimates of a finite element method for a class of NPEs, Li [17] introduced a space time adaptive finite element method continuous in space but discontinuous in time, Shi *et al.* [23, 24] developed nonconforming finite element methods, Tian and Wang [26] proved the Hölder continuity of the second derivatives $D_x^2 u$ for fully nonlinear, uniformly parabolic equations, Chatzipantelidis and Ginting [1] studied a finite volume discretisation in convex polygonal domains and established L^2 and H^1 error estimates, Li and Wang [16] determined unconditionally optimal error estimates of a linearised Crank-Nicolson Galerkin finite element methods for a strongly nonlinear parabolic system in R^2 and R^3, and Chen and Wang [8] studied an H^1-Galerkin mixed finite element method.

Discretising a semilinear elliptic boundary value problem and nonlinear elliptic partial differential equation, Xu [29, 30] introduced a two-grid finite element method. The main idea of this method consists in solving the problem on a coarse grid with a grid size H and then proceed with a linear problem on a fine grid of size $h \ll H$. The method was used by various researchers — e.g. Dawson and Wheeler [11, 12] studied a two-grid mixed finite element method and a two-grid finite difference method for NPEs, Chen *et al.* [9, 10] analysed a two-grid algorithm for NPEs by expanded mixed finite element methods, Wu and Allen [28] considered a two-grid method for mixed finite element solutions of reaction-diffusion equations, Liu *et al.* [20] investigated a two-grid algorithm for mixed finite element solution of NPEs, Zhang *et al.* [33, 34] developed a fully discrete two-grid finite-volume method and two-grid characteristic finite volume element methods for nonlinear parabolic problems, Yang [32] established errors of a two-grid discontinuous Galerkin method for NPEs, Wang *et al.* [27] studied a two-grid finite element method with Crank-Nicolson fully discrete scheme for the time-dependent Schrodinger equation, Chen *et al.* [5–7] considered two-grid finite volume element methods and two-grid finite element methods for NPEs in semidiscrete scheme. The authors of this work also studied a two-grid finite element method for a nonlinear Sobolev equation based on a backward Euler Galerkin scheme and a Crank-Nicolson Galerkin scheme — cf. [3, 4]. It seems that