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A Local Discontinuous Galerkin Method for Time-Fractional Burgers Equations

Wenping Yuan[1], Yanping Chen[2,*] and Yunqing Huang[1]

[1]*Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, Hunan, P.R. China.*
[2]*School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P.R. China.*

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Abstract. A local discontinuous Galerkin finite element method for a class of time-fractional Burgers equations is developed. In order to achieve a high order accuracy, the time-fractional Burgers equation is transformed into a first order system. The method is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. The scheme is proved to be unconditionally stable and in linear case it has convergence rate $\mathscr{O}(\tau^{2-\alpha} + h^{k+1})$, where $k \geq 0$ denotes the order of the basis functions used. Numerical examples demonstrate the efficiency and accuracy of the scheme.

AMS subject classifications: 65M60, 35K55

Key words: Time-fractional Burgers equation, Caputo fractional derivative, local discontinuous Galerkin method, stability, convergence.

1. Introduction

Fractional differential equations generally can be divided into three categories: time fractional equations [8, 14, 18, 38], space fractional equations [2, 13, 24, 36] and time-space fractional equations [1,16,26,35]. Analytical solutions of these equations are rarely known and even if they are, it is difficult to extend them to a general case. Therefore, it is important to study the properties of fractional differential equations and to develop numerical methods for the approximation of their solutions. Various numerical approaches to the equations with fractional derivatives include methods for equations with Caputo and Riemann-Liouville fractional derivatives. In particular, Sun and Wu [27] showed that the L^1-approximation formula for the order α Caputo fractional derivatives is of order $(2-\alpha)$. Lin and Xu [20] considered a spectral method with finite difference approximation for the time Caputo fractional diffusion equation. They proved that L^1 formula has the order

*Corresponding author. *Email addresses:* `yanpingchen@scnu.edu.cn` (Y. Chen), `ywp360428@163.com` (W. Yuan), `huangyq@xtu.edu.cn` (Y. Huang)

$(2-\alpha)$ and established a strict estimate. Lin *et al.* [19] studied a spectral method with L^1 scheme for fractional PDEs, Zhang *et al.* [38] combined finite difference methods with L^1-approximations, Li *et al.* [18] considered a finite element method and L^1 scheme for the time fractional Maxwell's equations, and Feng *et al.* [12] discussed a finite element method for a 2D multi-term time-fractional mixed sub-diffusion and diffusion-wave equations.

Classical Burgers equation is an important model in fluid dynamics [3], and fractional Burgers-type equations can be used to describe the cumulative effect of the wall friction through boundary layers. Therefore, considerable efforts have been spent on the development of numerical methods for such equations. El-Ajou [10] generalised the RPS method and obtained explicit and approximate solutions of the nonlinear fractional KdV-Burgers equation with time-space-fractional derivatives, Li *et al.* [17] developed a linear implicit finite difference scheme, which greatly reduces computational complexity, Mohebbi [25] considered a method based on a finite difference scheme in time and the Chebyshev spectral collocation method in space, Esen and Tasbozan [11] employed a finite element method based on the cubic B-spline collocation method. Other numerical methods for this type equations are studied in [21, 22, 28]. In this work, we consider a local discontinuous Galerkin (LDG) method for the following generalized time fractional Burgers equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+g(u)_x-d\frac{\partial^2 u(x,t)}{\partial x^2}=f(x,t),\quad (x,t)\in[a,b]\times[0,T], \tag{1.1}$$

where $0<\alpha\le 1$ and $d>0$ is a positive constant. The boundary conditions is either periodic or compact and the initial condition is

$$u(x,0)=u_0(x),\quad x\in[a,b].$$

The time fractional derivative $\partial^\alpha u/\partial t^\alpha$ in the Eq. (1.1) is the Caputo derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha},\quad 0<\alpha<1, \tag{1.2}$$

where Γ refers to the usual Gamma function. As α approaches 1^-, it is the first order integral derivation.

Local discontinuous Galerkin methods for time fractional partial differential equations have been recently studied in [15, 29–31]. In particular, it was shown that a fully discrete LDG method for the time-fractional KdV equation in [31] converges as $\mathscr{O}(\tau^{-\alpha}h^{k+1}+\tau^{2-\alpha}+\tau^{-\alpha/2}h^{k+1/2}+h^{k+1})$, where $k\ge 0$ is the order of the basis function, and τ and h are, respectively, time and space step sizes. In present work, we show that the order of convergence for the LDG method can be improved to $\mathscr{O}(\tau^{2-\alpha}+h^{k+1})$. On the other hand, Du *et al.* [9] introduced an LDG method with high-order temporal convergence rate for nonlinear time-fractional fourth-order PDEs, and Liu [23] considered an LDG method for a time-fractional subdiffusion equation with the third-order temporal convergence rate. The LDG method proposed by Cockburn and Shu [7] is an important class of numerical methods with a high-order accuracy. Due to discontinuity of the elements in the approximation space, it is well