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An Algebraic Multigrid Method for Eigenvalue Problems and Its Numerical Tests

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Abstract. In order to solve eigenvalue problems, an algebraic multigrid method based on a multilevel correction scheme and the algebraic multigrid method for linear equations is developed. The algebraic multigrid method setup procedure is used for construction of an hierarchy and intergrid transfer operators. In this approach, large scale eigenvalue problems are solved by algebraic multigrid smoothing steps in the hierarchy and by low-dimensional eigenvalue problems. The efficacy and flexibility of the method is demonstrated by a number of test examples and the global convergence, which does not depend on the number of eigenvalues wanted, is obtained.

AMS subject classifications: 65N30, 65N25, 65L15, 65B99

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1. Introduction

Algebraic multigrid (AMG) method was introduced by Brandt *et al.* [2] while investigating multigrid algorithms for automatic algorithm design. However, its convergence has

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been proved only for symmetric positive definite M-matrices with weak diagonal dominance [27] and in few other cases not involving M-matrices [16, 24, 36]. One of essential problems in AMG methods is closely connected to the choice of coarse grid and intergrid transfer operators. The problem attracted considerable attention and in addition to the classical coarsening strategy proposed by Ruge and Stüben [27], other approaches such as aggregation and smooth aggregation methods [25, 31], compatible relaxation [5, 22], and interpolation [6] and energy-based strategies [4] have been exploited. Cleary *et al.* [10] carried out numerical experiments to study the robustness and scalability of the AMG method. Parallel and adaptive AMG methods have also been studied in [7, 12]. The simplicity of the AMG method leads to its application to various problems — cf. Refs. [1, 11, 23].

In this work, we deal with the computation of q eigenpairs (maybe not of the smallest magnitude) for the following generalised eigenvalue problem: Find $(\lambda^{(j)}, u^{(j)}) \in \mathbb{R}\times\mathbb{R}^N$, $j = 1, 2, \ldots, q$ such that $(u^{(j)})^T M u^{(k)} = \delta_{jk}, j, k = 1, 2, \ldots, q$ and

$$Au^{(j)} = \lambda^{(j)} M u^{(j)}, \quad j = 1, 2, \ldots, q, \tag{1.1}$$

where A is a real symmetric positive definite $N \times N$ matrix and M a real symmetric semi-positive $N \times N$ matrix. Note that generalised eigenvalue problems (1.1) arise in the discretisation of the elliptic partial differential equations of electromagnetics, quantum chemistry, material, acoustic data science, and so on. These applications usually require high resolution results and, consequently, suitable discretisations of large scale algebraic eigenvalue problems. Therefore, the construction of efficient eigensolvers with a nearly optimal computational complexity is very important.

It is natural to use AMG and MG methods in eigenvalue problems [3, 8, 13, 14, 28, 35, 37]. A good survey of various application of the AMG methods in eigenvalue problems is presented in [17]. In these methods, an AMG strategy is adopted as the only solver in inner iterations combined with special outer iterations, such as inverse power, shift-and-inverse, Rayleigh-quotient, and locally optimal block preconditioned conjugate gradients. But the application of the AMG method does not lead to a new eigensolver (outer iteration). Recently, a new multilevel correction method has been proposed to solve eigenvalue problems [18–21, 26, 32–34]. The method is based on a new understanding of Aubin-Nitsche technique in the finite element method [19]. In contrast to the methods considered in [17], where AMG is used only as a preconditioner of the stiffness matrix, the coarse space of the multigrid method is employed to improve the working subspace in the eigenvalue problem solving [15]. Therefore, in this multilevel correction scheme, the solution of eigenvalue problem on the finest level mesh can be reduced to solving a sequence of standard boundary value problems on multilevel meshes and eigenvalue problems on a low-dimensional space. Hence, the computational work and the memory required can be at an optimal level. The above discussion shows that the application of a multigrid method to a multilevel correction scheme can provide a new eigensolver.

Motivated by the AMG method for boundary value problems and the multilevel correction method, we develop a new AMG method for eigenvalue problems. It can compute various eigenpairs (which may be not of the smallest magnitude) and allows a free choice of the eigensolvers for the low dimensional eigenvalue problems considered. With simple