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# The Riemann-Hilbert Approach and $N$-Soliton Solutions of a Four-Component Nonlinear Schrödinger Equation

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**Abstract.** A four-component nonlinear Schrödinger equation associated with a $5\times 5$ Lax pair is investigated. A spectral problem is analysed and the Jost functions are used in order to derive a Riemann-Hilbert problem connected with the equation under consideration. $N$-soliton solutions of the equation are obtained by solving the Riemann-Hilbert problem without reflection. For $N = 1$ and $N = 2$, the local structure and dynamic behavior of some special solutions is analysed by invoking their graphic representations.

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**Key words**: Four-component nonlinear Schrödinger equation, Riemann-Hilbert approach, $N$-soliton solutions.

## 1. Introduction

The nonlinear Schrödinger equation (NLS) is an important integrable model. It is closely related to nonlinear problems in theoretical physics such as nonlinear optics and ion acoustic waves of plasmas. On the other hand, higher-order coupled NLS equations are used to describe the effects of cubic-quintic nonlinearity, self-deepening, and self-frequency shifting. Among numerous solutions of these equations, soliton solutions play a crucial role in some complex nonlinear phenomena. At present, there are many methods to find the solutions of nonlinear integrable models — e.g. inverse scattering transform [1], Darboux transform [14], Hirota bilinear method [6], and Lie group method [2]. In particular, let us note the inverse scattering transform method, which is especially efficient in finding soliton solutions of the corresponding initial value problems. For second-order spectral problems, the inverse scattering theory is equivalent to Riemann-Hilbert (RH) approach. On the other hand, some of the higher-order spectral problems have to be transformed into RH problem.

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This approach, developed by Zakharov *et al* [34], was successively applied to various integrable systems with a single component [3–5,8–13,16–21,24–30,32,33,35–38]. However, to the best of authors' knowledge, only a few studies deal with multi-component problems.

The nonlinear Schrödinger equation has the form

$$i\frac{\partial \psi_\alpha}{\partial t} + \frac{\partial^2 \psi_\alpha}{\partial x^2} + \sum_{\beta,\gamma} \psi_\beta^* \Lambda_{\beta\gamma} \psi_\gamma \psi_\alpha = 0, \tag{1.1}$$

where $\Lambda$ is an Hermitian matrix [15]. A multi-component multisoliton solution of the Eq. (1.1) has been constructed by the Hirota's bilinearisation method — cf. [7]. The well-known general two-component coupled nonlinear Schrödinger equation — cf. [22],

$$\begin{aligned} &ip_t + p_{xx} + 2\big(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*\big)p = 0, \\ &iq_t + q_{xx} + 2\big(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*\big)q = 0, \end{aligned} \tag{1.2}$$

where $a, c$ are real constants, $b$ is a complex constant, and "$*$" denotes the complex conjugation, is a special case of the Eq. (1.1). In physics, $a$ and $c$ describe the self-phase modulation and cross-phase modulation effects, and $b$ and $b^*$ the four-wave mixing effects.

Furthermore, the three-component nonlinear Schrödinger equations has the form

$$\begin{aligned} &iq_{1t} + q_{1xx} - 2\big[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\mathrm{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)\big]q_1 = 0, \\ &iq_{2t} + q_{2xx} - 2\big[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\mathrm{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)\big]q_2 = 0, \\ &iq_{3t} + q_{3xx} - 2\big[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\mathrm{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)\big]q_3 = 0, \end{aligned} \tag{1.3}$$

where $a, c, f$ are real constants, $b, d, e$ complex constants, and "Re" denotes the real part. These equations are studied by extending the Fokas unified approach by Yan in [31]. In particular, it was shown that the Eq. (1.3) can be reduced to three-component NLS equations with various conditions on parameters $a, b, c, d, e$ and $f$. More precisely,

- If $a = c = f = -1$ and $b = d = e = 0$, the Eq. (1.3) reduces to a three-component focused NLS equation.
- If $a = c = f = 1$ and $b = d = e = 0$, the Eq. (1.3) reduces to a three-component defocused NLS equation.
- If $a = -1$, $c = f = 1$ and $b = d = e = 0$ or $a = 1$, $c = f = -1$ and $b = d = e = 0$, the Eq. (1.3) reduces to a three-component mixed NLS equation.
- For other choice of parameters, the Eq. (1.3) reduces to other three-component NLS equations.

In this work, we consider a four-component nonlinear Schrödinger (FCNLS) equation — viz.

$$iq_{1t} + q_{1xx} - 2\big[a_{11}|q_1|^2 + a_{22}|q_2|^2 + a_{33}|q_3|^2 + a_{44}|q_4|^2$$