East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.270520.210920

Vol. **11**, No. 1, pp. 164-180
February 2021

A Compact Difference Scheme for Time-Fractional Dirichlet Biharmonic Equation on Temporal Graded Meshes

Mingrong Cui*

School of Mathematics, Shandong University, Jinan 250100, Shandong, China.

Received 27 May 2020; Accepted (in revised version) 21 September 2020.

Abstract. The stability of a compact finite difference scheme on general nonuniform temporal meshes for a time fractional two-dimensional biharmonic problem is proved and graded mesh error estimates are derived. By using the Stephenson scheme for spatial derivatives discretisation, we simultaneously obtain approximate values of the gradient without any loss of accuracy. The discretisation of the Caputo derivative on graded meshes leads to a fully discrete implicit scheme. Numerical experiments support the theoretical findings and indicate that for problems with nonsmooth solutions, graded meshes have an advantage for very coarse temporal meshes.

AMS subject classifications: 65M06, 65M12, 65M15, 65N06, 65N22

Key words: Fractional biharmonic equation, nonsmooth solution, graded mesh, compact difference scheme, stability and convergence.

1. Introduction

Let $\Omega=(0,L)^2$ and Δ^2 be the biharmonic operator,

$$\begin{aligned}&\Delta^2 u(x,y,t)\\ :=&\frac{\partial^4 u}{\partial x^4}(x,y,t)+\frac{\partial^4 u}{\partial y^4}(x,y,t)+2\frac{\partial^4 u}{\partial x^2\partial y^2}(x,y,t),\quad (x,y)\in\Omega,\quad t\in(0,T].\end{aligned}$$

In this work we consider the time fractional equation

$$\begin{aligned}&{}_0^C D_t^\alpha u(x,y,t)+\Delta^2 u(x,y,t)=f(x,y,t), && (x,y)\in\Omega, && t\in(0,T],\\ &u(x,y,0)=g(x,y), && (x,y)\in\Omega, &&\\ &u(x,y,t)=0,\quad \frac{\partial u}{\partial \vec{n}}(x,y,t)=0, && (x,y)\in\partial\Omega, && t\in(0,T],\end{aligned}\tag{1.1}$$

*Corresponding author. *Email address:* `mrcui@sdu.edu.cn` (M.R. Cui)

where $\vec{n}$ is the unit outwards normal vector to the boundary $\partial\Omega$ of Ω and ${}_0^C D_t^\alpha$, $0<\alpha<1$ denotes the Caputo fractional derivative defined by

$$ {}_0^C D_t^\alpha u(x,y,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial \tau}(x,y,\tau) d\tau. \tag{1.2} $$

Fractional fourth-order partial differential equations arise in various applications [20]. We recall that fourth-order compact difference schemes in space, developed for one dimensional problems in [11, 24], have been extended to two dimensional problems in [7, 12]. In particular, the convergence of the methods was investigated under the assumption that the exact solution is smooth and the mesh is uniform both in time and space. On the other hand, the solutions of fractional partial differential equations (FPDEs) usually are non-smooth because the singularity of the time fractional derivative leads to a weak singularity near the initial time $t=0$. This may cause the loss of accuracy of the numerical method under consideration. Thus for the one-dimensional time-fractional diffusion equation considered in [18], Stynes and O'Riordan [23] showed that for all $(x,t)\in[0,L]\times(0,T]$, the derivatives of the corresponding solution u can be estimated as follows:

$$ \begin{aligned} \left|\frac{\partial^k u}{\partial x^k}(x,t)\right| &\le C && \text{for} \quad k=0,1,2,3,4, \\ \left|\frac{\partial^l u}{\partial t^l}(x,t)\right| &\le C\big(1+t^{\alpha-l}\big) && \text{for} \quad l=0,1,2. \end{aligned} $$

It was also pointed out that the typical solutions of time-fractional reaction-diffusion problem have an initial layer at $t=0$ and the derivative $(\partial u/\partial t)(x,t)$ blows up as $t\to 0^+$. Yuste and Quintana-Murillo [25] generalised $L1$ formula to non-uniform meshes. Zhang *et al.* [27] established the stability of the $L1$ approximations of the Caputo derivative on nonuniform meshes and proved the convergence estimate $\mathcal{O}(N^{\alpha-2}+h^4)$ for a special temporal mesh.

A concise survey of finite element methods for subdiffusion problems with nonsmooth data is given in [13], and finite difference methods for nonlinear fractional differential equations based on non-uniform meshes are presented in [15]. It is worth noting that for problems with nonsmooth solutions, graded temporal partitions are more suitable (see [17] where a sharp error estimate for non-uniform meshes are proved). A time-fractional Benjamin-Bona-Mahony equation and nonlinear Korteweg-de Vries equation are, respectively, discussed in [19] and [21]. Galerkin-Legendre spectral schemes for nonlinear time-space fractional diffusion-reaction equations studied by Zaky *et al.* [26], use the $L1$ scheme on graded meshes for approximation of the time fractional derivative. A compact ADI scheme for two-dimensional fractional sub-diffusion equation with Neumann boundary condition was given in [5] and time fractional Burgers' equations was discussed in [16].

Considering the fractional fourth-order problems (1.1), we note that the method of separation of variables shows that the solution also has a layer at the $t=0$. We will follow the ideas of [18] and discuss the details here. Note that since the Eq. (1.1) is linear, the