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Application of the Nonlinear Steepest Descent Method to the Coupled Sasa-Satsuma Equation

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Abstract. We use spectral analysis to reduce Cauchy problem for the coupled Sasa-Satsuma equation to a 5×5 matrix Riemann-Hilbert problem. The upper and lower triangular factorisations of the jump matrix and a decomposition of the vector-valued spectral function are given. Applying various transformations related to the Riemann-Hilbert problems and suitable decompositions of the jump contours and the nonlinear steepest descent method, we establish the long-time asymptotics of the problem.

AMS subject classifications: 35Q53, 35B40, 35Q55

Key words: Coupled Sasa-Satsuma equation, nonlinear steepest descent method, long-time asymptotics.

1. Introduction

The Sasa-Satsuma equation

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u\left(|u|^2\right)_x = 0 \tag{1.1}$$

also called the higher-order nonlinear Schrödinger equation, was originally aimed to describe the propagation of pulses in optical fiber [18, 19]. It attracted a considerable attention and has been extensively studied because of significant applications. The inverse scattering method [34] and the Hirota bilinear method [12] were used to obtain N-soliton solution of this equation. On the other hand, by linearising the corresponding spectral operator it was shown that the squared eigenfunctions of the Sasa-Satsuma equation can be represented as the sums of two terms, each of which is a product of Jost and adjoint Jost functions [43]. Akhmedieva *et al.* [2] studied the rogue wave spectra of the Eq. (1.1) and its presence in the spectra of chaotic wave fields produced by the modulation instability. Ling [22] obtained high order solution formulas in the determinant form by using a generalised Darboux transformation and the formal series method. In [44], finite genus solutions

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of the Sasa-Satsuma hierarchy, associated with a 3×3 matrix spectral problem, are obtained by using asymptotic expansions of the Baker-Akhiezer function and its Riemann theta function representation [37]. The Riemann-Hilbert approach, Darboux transformation and Riccati equation are employed in investigating the integrability of multi-coupled nonlinear integrable equations and finding their exact solutions — cf. Refs. [9,11,15,20,21,27,38,41].

Let

$$\mathscr{S}(\mathbb{R})=\left\{f(x)\in C^{\infty}(\mathbb{R}):\sup_{x\in\mathbb{R}}\left|x^{\alpha}\partial^{\beta}f(x)\right|<\infty,\forall\alpha,\beta\in\mathbb{N}\right\}$$

be the Schwartz class. In this work, we use the nonlinear steepest descent method in order to study the long-time asymptotic behavior of the Cauchy problem for the coupled Sasa-Satsuma equation

$$\begin{aligned}&u_t+u_{xxx}+6\left(|u|^2+|v|^2\right)u_x+3u\left(|u|^2+|v|^2\right)_x=0,\\&v_t+v_{xxx}+6\left(|u|^2+|v|^2\right)v_x+3v\left(|u|^2+|v|^2\right)_x=0,\\&u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\end{aligned}\tag{1.2}$$

where $u(x,t)$ and $v(x,t)$ are complex-valued potentials, $u_0(x),v_0(x)\in\mathscr{S}(\mathbb{R})$ and are generic in the sense that the below defined determinant $\det a(k)$ does not vanish in the lower complex half k-plane $\mathbb{C}_-$. The coupled Sasa-Satsuma equation can describe the simultaneous propagation in birefringent or two-mode fibers [32]. In [40], multi-soliton solutions of the coupled Sasa-Satsuma equation are derived by solving a Riemann-Hilbert problem. Besides, infiniteness of conserved quantities of the Eqs. (1.2) is discussed in [33], the Painlevé property in [36], and some other characteristics in [24,28,45]. The Deift-Zhou nonlinear steepest descent method introduced in [7] is aimed to study the long-time asymptotic behavior of solutions for the mKdV equation. The method was subsequently applied to a number of integrable nonlinear evolution equations associated with numerous matrix spectral problems [4–6, 8, 10, 13, 16, 17, 23, 25, 26, 29–31, 35, 42]. However, to the best of author's knowledge, the nonlinear steepest descent method has not been used in the study of long-time asymptotics for integrable equation associated with 5×5 matrix Lax pairs and the aim of this work is to extend the Deift-Zhou method to the Eqs. (1.2) associated with such Lax pairs. The main result of this paper is the following theorem.

Theorem 1.1. *Let $(u(x,t),v(x,t))$ be the solution for the Cauchy problem of the coupled Sasa-Satsuma equation* (1.2) *with $u_0(x)$ and $v_0(x)\in\mathscr{S}(\mathbb{R})$. If $x<0$ and $|x/t|\le C$, as $t\to\infty$, then the leading asymptotics of $(u(x,t),v(x,t))$ has the form*

$$\begin{aligned}&\big(u(x,t),v(x,t)\big)\\&=-\frac{\nu e^{\pi\nu/2}}{\sqrt{24tk_0\pi}}\Big[\delta_A^2e^{-\pi i/4}\Gamma(-i\nu)\big(\gamma_2(k_0),\gamma_4(k_0)\big)+\delta_A^{-2}e^{\pi i/4}\Gamma(i\nu)\big(\gamma_1^*(k_0),\gamma_3^*(k_0)\big)\Big]\\&\quad+\mathcal{O}\left(c(k_0)t^{-1}\log t\right),\end{aligned}$$

where C is a constant, Γ the Gamma function, $\gamma(k)=(\gamma_1(k),\gamma_2(k),\gamma_3(k),\gamma_4(k))$ the vector-