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An hp-Version of C^0-Continuous Petrov-Galerkin Time-Stepping Method for Second-Order Volterra Integro-Differential Equations with Weakly Singular Kernels

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Abstract. An hp-version of C^0-CPG time-stepping method for second-order Volterra integro-differential equations with weakly singular kernels is studied. In contrast to the methods reducing second-order problems to first-order systems, here the CG and DG methodologies are combined to directly discretise the second-order derivative. An a priori error estimate in the H^1-norm, fully explicit with respect to the local discretisation and regularity parameters, is derived. It is shown that for analytic solutions with start-up singularities, exponential rates of convergence can be achieved by using geometrically refined time steps and linearly increasing approximation orders. Theoretical results are illustrated by numerical examples.

AMS subject classifications: 65R20, 65M60, 65M15

Key words: hp-version, second-order Volterra integro-differential equation, weakly singular kernel, continuous Petrov-Galerkin method, exponential convergence.

1. Introduction

Let T and $\alpha \in [0,1)$ be real numbers, $I := (0,T]$ and $D := \{(t,s) : 0 \le s \le t \le T\}$. In this work, we study numerical methods for the following linear second-order Volterra integro-differential equations (VIDEs):

$$\begin{aligned} &u''(t) = p(t)u'(t) + q(t)u(t) + f(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u(s)ds, \quad t \in I, \\ &u(0) = u_0, \quad u'(0) = u_1. \end{aligned} \tag{1.1}$$

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The functions $p, q, f : I \to \mathbb{R}$ and $K(t,s)\colon D \to \mathbb{R}$ are assumed to be continuous on their respective domains. If $0 < \alpha < 1$, the Eq. (1.1) is referred to as the weakly singular VIDE. Hereafter, sometimes we will write $\dot{u}$ and $\ddot{u}$ for u' and u'', respectively.

Equations of the form (1.1) arise in various areas of physics and engineering — cf. [9] and the references therein. In the last few decades, the numerical analysis of VIDEs attracted considerable attention and the list of numerical methods developed includes collocation [5, 14, 20, 28], Runge-Kutta [4, 38], continuous and discontinuous Galerkin methods [21, 23]. We also refer the reader to the monographs [6, 22]. However, to the best of authors' knowledge, most of the methods deal with VIDEs of the first-order. For the second-order VIDEs, numerical approaches are not well studied and are mainly restricted to collocation methods [1, 7, 29, 32].

It is well-known that the solutions of integral and integro-differential equations of Volterra type with weakly singular kernels are generally not smooth at the initial point [6]. Such singular behavior may result in a low convergence rate of the corresponding numerical method even if high order polynomials are used. In order to overcome the problems generated by the solution singularities, a number of special approaches such as collocation method with graded meshes [5], nonpolynomial spline collocation method [3], and hybrid collocation method [11] were developed. These methods are mainly based on the h-version approach with diminishing time steps and polynomials of a fixed order. Therefore, the best possible convergence order can be only algebraic. In contrast, the p- and hp-version approaches employ approximation polynomials of various order. In particular, since hp-version methods allow locally varying mesh sizes and approximation orders, smooth solutions with possible local singularities can be approximated with a high algebraic order or even with exponential convergence rate [26].

In recent years, p- and hp-versions of Galerkin finite element methods are widely used in approximations of VIDEs. For example, an hp-version of the discontinuous Galerkin time-stepping method for the first-order VIDEs and parabolic VIDEs is, respectively, studied in [8, 24], an hp-version of the continuous Petrov-Galerkin method for the first-order linear and nonlinear VIDEs is considered in [35–37], and hp-versions of the discontinuous and continuous Galerkin methods for nonlinear initial value problems are discussed in [25, 33, 34]. Some other high-order methods such as spectral Galerkin and collocation methods have been also applied to Volterra type equations — cf. Refs. [10, 12, 13, 15, 17, 19, 27, 31, 32]. However, the hp-methods for the second-order VIDEs are not well studied and so far, to the best of our knowledge, the problem mentioned has been studied only in [18] for the equations with smooth kernels.

The present work extends the approach of [18] to the second-order VIDE (1.1) with weakly singular kernels. In the method under consideration, the trial spaces consist of C^0-continuous piecewise polynomials, whereas test spaces use discontinuous piecewise polynomials. At each time step, the formulation can be decoupled into local problems, so that the method can be viewed as a time-stepping scheme. Such a C^0-CPG time-stepping method has been before employed in time discretisation of the second-order linear evolution problems [16, 30], but the error analysis is based on the traditional h-version approach. We provide certain local time steps conditions, which ensure the well-posedness of the hp-