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Efficient Hermite Spectral Methods for Space Tempered Fractional Diffusion Equations

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Abstract. Spectral and spectral collocation methods for tempered fractional diffusion equations on the real line $\mathbb{R}$ are developed. Applying the Fourier transform to the problem under consideration, we reduce it to systems of algebraic equations. Since Hermite functions are the eigenfunctions of the Fourier transform, they are used in the construction of spectral and spectral collocation methods for the algebraic equations obtained. The stability and convergence of the methods are studied. Numerical examples demonstrate the efficiency of the algorithms and confirm theoretical findings.

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1. Introduction

Fractional integrals and derivatives are widely used to model anomalous phenomena arising in physics [1, 10, 12, 25, 32–34], finance [19, 29, 37], biology [3, 21], and hydrology [2, 13, 17, 31]. A prominent example of such applications is given by the anomalous diffusion equation

$$\partial_t^{\beta} p(x,t) = \partial_x^{\alpha} p(x,t),$$

where $p(x,t)$ is the probability density function and $0 < \beta < 1$, $0 < \alpha < 2$ [32]. Since the spatial fractional derivative causes the asymptotic decay $|x|^{-1-\alpha}$ of the solution, the

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moment conditions of the traditional diffusion equation are violated. There are various approaches to fix the problem — cf. [8, 22, 23, 26, 27, 39, 44], including the use of the exponential tempering factor $e^{-\lambda|x|}, \lambda > 0$ in the particle jump density function. This leads to the tempered fractional diffusion equation, which finds numerous applications [4–6, 15, 30, 35, 36].

In order to find the solution of a tempered fractional diffusion equation, numerical methods based on local operations have been developed recently. In particular, Li and Deng [24] considered a high order difference scheme for equations on bounded domains, Sabzikar *et al.* [36] introduced a finite difference method on truncated domains, Baeumera and Meerschaert [4] provided finite difference and particle tracking methods, Cartea and del-Castillo-Negrete [6] constructed a finite difference scheme for a Black-Merton-Scholes model with tempered fractional derivatives, Sun *et al.* [40] applied different methods to spatial operators in the time-space tempered fractional Fokker-Planck equation on a finite domain, Deng and Zhang [16] designed finite difference and finite element schemes to simulate the backward time tempered fractional Feynman-Kac equation, Dehghan and Abbaszadeh [14] employed finite element methods to the space fractional tempered diffusion-wave equation, Chen and Deng [9] developed an unconditionally stable second-order finite difference scheme for the space-time tempered fractional diffusion-wave equation, Çelik and Duman [7] proposed a Galerkin finite element method for symmetric tempered fractional diffusion equations. Nevertheless, since the tempered derivative operators are global, the spectral methods using global bases are well-suited for solving tempered fractional diffusion equations. Hanert and Piret [20] employed a pseudospectral method based on the Chebyshev polynomial expansion for solving space and time tempered fractional diffusion equation. Zayernouri *et al.* [43] defined tempered Jacobi poly-fractonomials and used them to simulate tempered fractional differential operators. All the above mentioned numerical methods are applied to tempered fractional derivatives in the bounded domain. However, since there are many problems with tempered fractional derivatives related to random walks on the whole line, it is important to study the corresponding equations on unbounded domains. Nevertheless, there are only a few works devoted to such problems.

In this paper, we consider the following space tempered fractional diffusion equation on the real line — cf. [5, 36]: For $\mu \in (k-1, k)$, $k = 1, 2$,

$$\begin{aligned}
&\partial_t u(x,t) + (-1)^{k+1}\left(p\partial_{+,x}^{\mu,\lambda} + q\partial_{-,x}^{\mu,\lambda}\right)u(x,t) = f(x,t), \quad x \in \mathbb{R}, \quad 0 < t \leq T, \\
&u(x,0) = u_0(x),
\end{aligned}$$

where $p + q = 1$, $p, q \geq 0$, and $\partial_{+,x}^{\mu,\lambda}$, $\partial_{-,x}^{\mu,\lambda}$ are the tempered fractional derivatives defined by (2.5)-(2.6). Chen *et al.* [11] developed an efficient spectral method for solving this equation. Thus they introduced a family of generalised Laguerre functions and derived useful formulas for tempered fractional integrals and derivatives. However, the numerical implementation of their method requires substantial efforts because of the complicated definition of the tempered fractional derivatives. Recently, Mao and Shen [28] showed that the Hermite functions are the eigenfunction of the Fourier transform operator — Lemma 2.3 below. This can be used to reduce complicated non-local fractional problem to simple