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A Superconvergent Nonconforming Mixed FEM for Multi-Term Time-Fractional Mixed Diffusion and Diffusion-Wave Equations with Variable Coefficients

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Abstract. An unconditionally stable fully-discrete scheme on regular and anisotropic meshes for multi-term time-fractional mixed diffusion and diffusion-wave equations (TFMDDWEs) with variable coefficients is developed. The approach is based on a non-conforming mixed finite element method (FEM) in space and classical $L1$ time-stepping method combined with the Crank-Nicolson scheme in time. Then, the unconditionally stability analysis of the fully-discrete scheme is presented. The convergence for the original variable u and the flux $\vec{p} = \mu(\mathbf{x})\nabla u$, respectively, in H^1- and L^2-norms is derived by using the relationship between the projection operator R_h and the interpolation operator I_h. Interpolation postprocessing technique is used to establish superconvergence results. Finally, numerical tests are provided to demonstrate the theoretical analysis.

AMS subject classifications: 65M10, 78A48

Key words: Nonconforming mixed FEM, multi-term time-fractional mixed diffusion and diffusion-wave equations, $L1$ time-stepping method, Crank-Nicolson scheme, convergence and superconvergence.

1. Introduction

In recent years, fractional partial differential equations (FPDEs) have attracted a considerable attention. In contrast to integer-order partial differential equations, FPDEs have numerous advantages in applications, including hereditary processes in biology, physics, medicine, finance, fluid mechanics, environmental science, and water waves — cf. Refs. [7,

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9, 10, 12, 14, 18, 22, 26, 27, 54]. For example, in physics and materials one often uses time-fractional partial differential equations (TFPDEs) such as time-fractional diffusion equations (TFDEs) and time-fractional wave equations (TFWEs). Nevertheless, the use of only TFDEs or only TFWEs does not allow to describe certain processes accurately. In this paper, we focus on a nonconforming mixed FEM for a kind of TFPDEs with variable coefficients, called multi-term TFMDDWEs — viz.

$$\begin{aligned} & u_t + P(D_t^{\alpha})u(\mathbf{x},t) + P\left(D_t^{\beta}\right)u(\mathbf{x},t) \\ & \quad - \nabla\cdot(\mu(\mathbf{x})\nabla u(\mathbf{x},t)) = f(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T], \\ & u(\mathbf{x},t) = 0, \qquad\qquad (\mathbf{x},t)\in\partial\Omega\times(0,T], \\ & u(\mathbf{x},0) = u_0(\mathbf{x}), u_t(\mathbf{x},0) = \tilde{u}_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset R^2$ is a bounded convex polygonal region with the boundary $\partial\Omega$, $\mathbf{x}=(x,y)$, $u_0(\mathbf{x})$, $\tilde{u}_0(\mathbf{x})$, $\mu(\mathbf{x})$, $f(\mathbf{x},t)$ are sufficiently smooth functions and there are constants μ_1,μ_2 such $0<\mu_1\le\mu(\mathbf{x})\le\mu_2$. The operators $P(D_t^{\alpha}), P(D_t^{\beta})$ are defined by

$$P\left(D_t^{\alpha}\right) = D_t^{\alpha} + \sum_{i=1}^{r} l_i D_t^{\alpha_i}, \quad l_i>0, \quad r\in\mathbb{N}^+, \quad 0<\alpha_1<\alpha_2<\cdots<\alpha_r<\alpha<1,$$

$$P\left(D_t^{\beta}\right) = D_t^{\beta} + \sum_{j=1}^{m} \omega_j D_t^{\beta_j}, \quad \omega_j>0, \quad m\in\mathbb{N}^+, \quad 1<\beta_1<\beta_2<\cdots<\beta_m<\beta<2,$$

where D_t^{γ} is the left-sided Caputo fractional derivative of order γ with respect to t, i.e.

$$D_t^{\gamma}u(\mathbf{x},t) := \begin{cases} \dfrac{1}{\Gamma(n-\gamma)}\displaystyle\int_0^t (t-s)^{n-1-\gamma}\frac{\partial^n u(\mathbf{x},s)}{\partial s^n}ds, & \gamma\notin\mathbb{N}^+, \\ \dfrac{\partial^{\gamma}u(\mathbf{x},t)}{\partial t^{\gamma}}, & \gamma\in\mathbb{N}^+ \end{cases}$$

with Γ denoting the Gamma function — cf. [22].

Analytical solutions of the most TFDEs and TFWEs are not known. Thus numerical methods for solving TFDEs and TFWEs have attracted increasing attention. For example, Ammi *et al.* [2] studied a fully discrete scheme for TFDEs based on finite difference method in time and FEM in space. In [42], a finite difference method was used to solve a modified TFDE, and the unconditional stability and an optimal-order error estimate of the method were discussed. An efficient Petrov-Galerkin spectral method for a TFDE with its error analysis was presented in [31]. Galerkin methods are a popular approach for solving TF-PDEs — e.g. Sun *et al.* [39] proposed a semi-discrete scheme with a local discontinuous Galerkin discretisation in the spatial variable. Wang and Ren [43] considered a high-order compact finite difference method and numerical analysis for a single TFDE. Meshless methods are also widely used. In particular, Kumar *et al.* [15] studied a radial basis meshless local collocation method. In order to obtain a discrete scheme for time fractional diffusion-wave equation, Liu *et al.* [25] developed a novel finite difference method. Soori and Aminataei [37] presented the stability of a sixth-order compact alternating direction implicit