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A Second Order Numerical Scheme for Fractional Option Pricing Models

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Abstract. A number of fractional option models (FMLS, CGMY, KoBol) are proposed and studied under assumption that the motion of the underlying assets follows a Lévy process. Numerical methods for these option pricing models are based on solution of fractional partial differential equations. To discretise them, we employ a second order numerical scheme and study its stability and convergence. Numerical experiments show the efficiency of the method and its convergence. Simulations related to practical stock markets, further confirms the robustness of the scheme and shows that KoBol model has advantage over the classical Black-Scholes model.

AMS subject classifications: 65M06, 65M12, 65M32, 91G60

Key words: Lévy process, fractional partial differential equation, option pricing, finite difference, stock index option.

1. Introduction

The problem of pricing financial derivatives is a very important topic in financial engineering. Black-Scholes (BS) model [2] is the earliest systematic treatment of this problem and Merton [26] improved the model by introducing a jump process and the stochastic volatility [25]. According to the BS model, the price of an option $V(S,t)$ satisfies the following partial differential equation (PDE)

$$\frac{\partial V(S,t)}{\partial t}+rS\frac{\partial V(S,t)}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V(S,t)}{\partial S^2}=rV(S,t), \tag{1.1}$$

where r is the risk-free interest rate and $\sigma\,(\geqslant 0)$ is the volatility of the returns from the underlying asset price S. Setting $x=\ln S$, one rewrites the partial differential equation (1.1) as following advection-diffusion equation with constant coefficients

$$\frac{\partial V(x,t)}{\partial t}+\left(r-\frac{1}{2}\sigma^2\right)\frac{\partial V(x,t)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V(x,t)}{\partial x^2}=rV(x,t).$$

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The Black-Scholes model attracted a considerable attention and is widely used in financial engineering because of its simplicity and clarity. However, the assumptions of frictionless trade, smooth price change and the market completeness of the BS model are too strong to use them in practical markets. Thus the volatility in the equation is strongly related to the option maturity and strike, which leads to the volatility smile phenomenon in practical market [15].

Numerous attempts have been made to adjust the Black-Scholes model to practical markets requirements. Merton [26] proposed a jump-diffusion model, where jump process obeys normal distribution law. Kou [18] assumes that the asset price jump follows a log-double-exponential distribution and constructed the corresponding jump-diffusion model. Hull and White [14] and Heston [13] proposed stochastic volatility models. In addition, Davis *et al.* [9] and Amster *et al.* [1] studied these models with transaction fees. There are various numerical methods developed for solving option models — e.g. finite difference method [19, 30], finite volume method [10–12], Laplace transform method [20] and a projected triangular decomposition method [33, 34].

On the other hand, Gaussian processes do not explain such empirical facts of financial markets as skewed and fat tailed distribution or large jump over small time steps. In particular, Mandelbrot [21] noted that the distribution of relative stock prices has a long tail and proposed to use an exponential non-normal Lévy process. Carr and Wu [5] considered an FMLS process which can capture the highly skewed feature of the implied density for log returns. Koponen [17] and Boyarchenko and Levendorskii [3] used a modified Lévy-α-stable process to simulate the dynamics of stocks, which is called KoBol process. Carr *et al.* [4] introduced a CGMY process which can capture the size and frequency of positive and negative jumps in the asset price movements.

These Lévy process based models attracted attention of many researchers in both applied mathematics and numerical analysis. Cartea and del-Castillo-Negrete [6] represented FMLS, CGMY and KoBol option pricing models as fractional partial differential equations (FPDE) and used the shifted Grünwald difference (SGD) formula for their solution. Marom and Momoniat [22] compared numerical results for three fractional option pricing models but did not prove the stability and convergence of the numerical scheme used. Using the Caputo fractional derivative, Chen [7] constructed a finite difference second order scheme for a one-sided FMLS model. Wang *et al.* [29] and Meng *et al.* [24] proposed preconditioned methods for two-sided FPDE with shifted Grünwald difference formula. Zhang *et al.* [31, 32] discretised the one-side FMLS model with second order accuracy and constructed a second order numerical scheme for tempered FPDE. Numerical experiments confirms the convergence of one-sided FPDE. Zhou *et al.* [35, 36] developed Laplace transform based methods for FPDE arising in the option pricing problem and analysed the second, order convergence rate in space.

In this paper, we consider a second order numerical scheme for a class of two-sided fractional partial differential equation based on the weighted and shifted Grünwald difference (WSGD) formula. The stability and second order convergence of the numerical scheme are analysed in details. In order to verify the convergence rate of the numerical scheme, we carry out a number of numerical experiments. Furthermore, we use the data of China A