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High-Order Non-Conservative Simulation of Hyperbolic Moment Models in Partially-Conservative Form

J. Koellermeier[1,2,*] and M.J. Castro[3]

[1] *Institut für Mathematik, Freie Universität Berlin, Germany.*
[2] *School of Mathematical Sciences, Peking University, China.*
[3] *Departamento de Analisis Matematico, Universidad de Malaga, Spain.*

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Abstract. In this paper the first dedicated study on high-order non-conservative numerical schemes for hyperbolic moment models is presented. The implementation uses a new formulation that allows for explicit evaluation of the model while satisfying conservation of mass, momentum, and energy. The high-order numerical schemes use a path-conservative treatment of the non-conservative terms and a new consistent evaluation of the eigenvalues. The numerical results of two initial value problems, one stationary test case and a boundary value problem, yield stable and accurate solutions with convergence towards the reference solution despite the presence of a non-conservative term. A large speedup or accuracy gain in comparison to existing first-order codes could be demonstrated.

AMS subject classifications: 65M08, 76P05

Key words: Hyperbolic moment model, non-conservative, high-order scheme.

1. Introduction

Moment models yield efficient discretisations of kinetic equations in the microscopic velocity space [51]. In addition to their application for modeling of rarefied gases, they have been applied to a number of other fields, for example, in swarming models, radiotherapy and shallow flows [20, 24, 29, 36].

The development of moment models has begun with the work of Grad [25], but the resulting equations lacked hyperbolicity, such that further work was merely focused on different approaches, like the maximum entropy moment method — cf. [38, 40], or special distribution functions [50]. Recently there has been a lot of work on mitigation of hyperbolicity problems and several new globally hyperbolic moment models have been proposed. Among them are the Hyperbolic Moment Equations (HME) [4] and the Quadrature-Based

*Corresponding author. *Email addresses:* koellermeier@pku.edu.cn (J. Koellermeier), mjcastro@uma.es (M.J. Castro)

Moment Equations (QBME) [30]. The underlying analysis of the hyperbolicity regularisation was given in [6, 23]. The hyperbolic models have been extended to the respective multi-dimensional cases in [5,31], with a rotationally invariant version of the QBME model in [23] and were widely applied [19, 26, 35].

Non-conservative terms are present in all of the new hyperbolic models. This leads to problems, as standard finite-volume schemes cannot be used and dedicated numerical methods have to be employed. It is widely known that non-conservative schemes can converge to different solutions if the non-conservative products are approximated in the wrong way [1,13]. There are, however, a lot of schemes available in the literature for the accurate solution of non-conservative, hyperbolic PDEs [10, 15, 37, 42, 43, 47]. In [34], a detailed study on the performance of first order numerical schemes for the non-conservative hyperbolic moment models was conducted and the results showed that a stable solution is possible despite the numerical difficulties.

An investigation of a second order scheme, which was applied on top of the projection algorithm introduced in [8], was carried out in [9]. But, to the best of our knowledge, no detailed investigation of higher-order schemes — i.e. third-order ones, was performed so far. Higher-order non-conservative schemes are described in [11, 12, 14]. It is particularly unclear whether the aforementioned schemes encounter problems with the hyperbolic moment models, as instabilities in the numerical solution might emerge from the fact that the high-resolution schemes reduce the amount of diffusion of first-order schemes.

In this paper, we investigate the solution of hyperbolic moment models by high-order non-conservative schemes in one-dimensional space. Different from [4, 9, 34], we reformulate the system of equations and give explicit formulations of the one-dimensional hyperbolic models for two new sets of variables that allow for exact conservation of mass, momentum and energy. The new formulation also makes it possible to explicitly evaluate the eigenvalues of the matrix in the new variables to reduce runtime. The numerical results for four different test cases are presented: a shock tube test case, a symmetric two beam test case, a stationary shock structure test case, and a Fourier flow test case. The tests focus on the investigation of the non-conservative behaviour and the differences in runtime and accuracy of the higher-order schemes. We use the first, second, and third order versions of the HLL, FORCE and MUSTA-FORCE schemes described in [14]. We compare speedup and accuracy gain with respect to other existing codes for the solution of hyperbolic moment models to assess the computational efficiency of the solution methods.

The rest of this paper is structured as follows. In Section 2, the hyperbolic moment equations are reformulated in two new variable sets that will allow for an efficient implementation. Section 3 gives a concise overview over the numerical schemes and explains the treatment of the non-conservative products. The results of all four test cases are presented and discussed in Section 4. The paper ends with a short conclusion.

2. Hyperbolic Moment Equations

In this section, we will reformulate the hyperbolic moment equations introduced in [4, 30] for two different sets of variables that will allow for exact conservation of mass,