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A High-Order Efficient Optimised Global Hybrid Method for Singular Two-Point Boundary Value Problems

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Abstract. An optimised global hybrid block method for second order singular boundary value problems with two boundary conditions is developed. A special attention is paid to the problems having solutions with singularities at the left end of the interval considered. The method is a combination of the optimised hybrid formulas in [43] and a new set of formulas. The ad hoc procedure is used just to pass the singularity and the main formulas are applied to obtain approximations at other discrete points. Numerical experiments show that the method is a good alternative for the problems studied.

AMS subject classifications: 65L10, 65L10

Key words: Singular boundary value problem, hybrid methods, block methods, optimisation technique.

1. Introduction

Two-point boundary value problems occur in various applications, including fluid flow, shock waves and geophysical models. The problems can be categorised as singular and singularly perturbed ones and we refer the reader to [4] for more information about BVPs. In the present work we are concerned with numerical solution of two-point singular boundary value problems (SBVPs) for ODEs. Such problems frequently occur in practical phenomena such as reaction-diffusion processes, chemical kinetics, physiological processes, thermal-explosion theory, electro hydro-dynamics and shallow membrane caps theory [5, 9, 11, 13, 15, 16, 18, 23]. Since it is not always possible to find closed form solutions of SBVPs, these

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problems are usually tackled numerically. Second order two-point SBVPs have a great scientific significance, so that they attracted attention of many researchers. Here we consider the singular boundary value problem

$$z''(x) = f\big(x, z(x), z'(x)\big), \quad x \in [a, b], \tag{1.1}$$

subject to one of the following types of boundary conditions (BCs):

$$\begin{aligned} &\text{Dirichlet: } z(a) = z_a,\ z(b) = z_b, \\ &\text{Neumann: } z'(a) = z'_a,\ z'(b) = z'_b \text{ or} \\ &\text{Mixed: } g_1\big(z(a), z'(a)\big) = \nu_a,\ g_2\big(z(b), z'(b)\big) = \nu_b. \end{aligned} \tag{1.2}$$

We also assume that the function f in (1.1) has a singularity at the left end of the integration interval — i.e. at the point $x = a$. Different codes have appeared in the literature in order to numerically deal with special cases of SBVPs. Thus, Russell and Shampine [39] presented various numerical methods for solving SBVPs, Roul *et al.* proposed a high-order numerical scheme based on a quartic B-spline optimal collocation method for nonlinear SBVPs, Pandey *et al.* [27,28] considered second and fourth order finite difference methods, and Abukhaled [1] employed a second order B-spline collocation scheme for special SBVPs. Other approaches to solving SBVPs are discussed in [5,9,11–23,25–28,35–40,44].

The present work deals with the development and analysis of a method combining two approaches — viz. hybrid and block methods specifically used in numerical integrators of the initial value problems for ODEs [10,24]. For more details on hybrid and block methods for solving different types of differential equations one can consult [6,14,30–34,42] and references therein. The paper at hand, is an extension of our earlier work. More precisely, we combine an optimised hybrid block method for second order ODEs studied in [43] with an ad hoc set of formulas used to treat singularities at the left end of the integration intervals.

The subsequent sections are as follows. In Section 2, the main and an ad hoc formulas are presented. Section 3 deals with the convergence of the method. In Section 4, we discuss the implementation of the method. Numerical experiments are carried out in Section 5, and concluding remarks are given in Section 6.

2. Main and Ad Hoc Formulas

Here we present the main formulas and an ad hoc strategy for the SBVP (1.1).

2.1. Main formulas

In order to derive the main formulas, we discretise the interval $[a, b]$ as

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b,$$

where $x_n = x_0 + nh$, $n = 0, 1, \ldots, N$ and $h = x_{n+1} - x_n$ is the stepsize. Let z_n be an approximation of the true solution $z(x)$ at x_n, i.e. $z_n \approx z(x_n)$. On the two-step block interval