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A Fast Temporal Second-Order Compact ADI Scheme for Time Fractional Mixed Diffusion-Wave Equations

Rui-Lian Du and Zhi-Zhong Sun*

School of Mathematics, Southeast University, Nanjing 211189, P.R. China.

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Abstract. A fast temporal second-order compact alternating direction implicit (ADI) difference scheme is proposed and analysed for 2D time fractional mixed diffusion-wave equations. The time fractional operators are approximated by mixed fast $L2\text{-}1_\sigma$ and fast $L1$-type formulas derived by using the sum-of-exponentials technique. The spatial derivatives are approximated by the fourth-order compact difference operator, which can be implemented by an ADI approach with relatively low computational cost. The resulting fast algorithm is computationally efficient in long-time simulations since the computational cost is significantly reduced. Numerical experiments confirm the effectiveness of the algorithm and theoretical analysis.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Time fractional mixed diffusion-wave equations, SOEs technique, ADI difference scheme, stability, convergence.

1. Introduction

The equations with nonlocal operators, such as fractional partial differential equations (FPDEs), are often used to simulate various problems in real world applications [1, 28] but their analytical solutions are rarely available. Therefore, efficient numerical methods for their solution are required. The list of numerical methods for FPDEs and time FPDEs (tFPDEs) contains $L1$ scheme [19,33], $L1$-2 scheme [12], $L2\text{-}1_\sigma$ scheme [2] and $L2$ scheme [26]. Other numerical schemes are considered in [25, 32, 35] and [6, 17, 20] in the cases of smooth and non-smooth solutions, respectively. The $L2\text{-}1_\sigma$ scheme is, probably, the most efficient because of its second-order accuracy and excellent properties of the discrete kernels crucial for theoretical analysis [2]. The $L2\text{-}1_\sigma$ formula is based on the quadratic interpolation of the front time layer and the linear interpolation of the current time layer. Overall, it achieves $(3-\alpha)$-order accuracy for the time fractional Caputo derivative of order α, $0 < \alpha < 1$.

*Corresponding author. *Email addresses:* drl158@163.com (R.-L. Du), zzsun@seu.edu.cn (Z.-Z. Sun)

It is well-known that the distinctive characteristic of fractional derivative is its intrinsically nonlocal property and historical dependence. Therefore, traditional numerical methods are time consuming, especially in high dimensional problems. In order to overcome this difficulties, McLean *et al.* [27] developed a fast summation algorithm for an evolution equation with memory, and Ke *et al.* [16] applied the fast Fourier transform to block triangular Toeplitz-like systems arising in the solution of tFPDEs. Another fast algorithm for computing fractional derivatives is proposed in [3]. Recently, Jiang *et al.* [15] introduced a fast $L1$ formula based on the sum-of-exponentials (SOEs) approximation. This reduces the computational cost from $\mathscr{O}(N^2)$ to $\mathscr{O}(N\log N)$ and the storage from $\mathscr{O}(N)$ to $\mathscr{O}(\log N)$, where N is the number of grid points in time interval. Besides, Yan *et al.* [34] proposed a fast $L2$-1_σ formula and investigated the stability and the convergence of the resulting scheme for fractional sub-diffusion problems, and Gao *et al.* [13] developed a fast $L2$-1_σ algorithm for multi-term tFPDEs and proved the unconditional stability and the second-order accuracy of the corresponding scheme.

Motivated by the results mentioned, in this work we introduce and analyse computationally efficient and stable approximations for a nonlocal model — viz. for the $2D$ time fractional mixed diffusion-wave equation (tFMDWE)

$$
\begin{aligned}
&{}_0^CD_t^\beta u(x,y,t) + {}_0^CD_t^\alpha u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), && (x,y)\in\Omega, \quad t\in(0,T], && (1.1)\\
&u(x,y,0) = \phi(x,y), \quad u_t(x,y,0) = \psi(x,y), && (x,y)\in\Omega, && (1.2)\\
&u(x,y,t) = 0, && (x,y)\in\partial\Omega, \;\; t\in[0,T], && (1.3)\\
&\phi(x,y)|_{(x,y)\in\partial\Omega} = 0, \quad \psi(x,y)|_{(x,y)\in\partial\Omega} = 0,
\end{aligned}
$$

where $\Omega := (0,L_1)\times(0,L_2)$, $\partial\Omega$ is the boundary of Ω, f is a given function, and $\Delta u = \partial_x^2 u + \partial_y^2 u$. Besides, ${}_0^CD_t^\alpha u(x,y,t)$ and ${}_0^CD_t^\beta u(x,y,t)$ denote the time Caputo derivatives of $u(x,y,t)$ with respect to t, i.e.

$$
\begin{aligned}
{}_0^CD_t^\alpha u(x,y,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,\xi)}{\partial\xi}\frac{d\xi}{(t-\xi)^\alpha}, && \alpha\in(0,1),\\
{}_0^CD_t^\beta u(x,y,t) &= \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{\partial^2 u(x,y,\xi)}{\partial\xi^2}\frac{d\xi}{(t-\xi)^{\beta-1}}, && \beta\in(1,2),
\end{aligned}
$$

and we use the notation $\bar\Omega = \Omega\cup\partial\Omega$ in what follows.

To the best of our knowledge, numerical methods for tFMDWE are not yet well studied, although various types of the Eq. (1.1) are widely used in real-life problems. Hao *et al.* [14] developed an $L2$ formula combined with a compact difference scheme for multi-term tFMDWE and proved its first order accuracy in time and fourth order accuracy in space. Ezz-Eldien *et al.* [8] presented a numerical method based on shifted Legendre polynomials and time-space spectral collocation method. For multi-term tFMDWEs with variable coefficients, Fan *et al.* [9] introduced an unconditionally stable fully-discrete scheme based on a nonconforming mixed finite element method in space and $L1$-CN method. Feng *et al.* [10] considered a more general version of tFMDWEs with a special time-space coupled derivative. This feature enables to consider the generalised Oldroyd-B fluid model, which is the