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Efficient and Accurate Computation of the Bogoliubov-De Gennes Excitations for the Quasi-2D Dipolar Bose-Einstein Condensates

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Abstract. An efficient spectrally accurate multigrid method for the Bogoliubov-de Gennes excitations of the quasi-2D dipolar Bose-Einstein condensates is proposed. The wave functions/eigenmodes are spatially discretised by the Fourier spectral method. The convolution-type nonlocal potentials are computed in $\mathcal{O}(N\log(N))$ operations with a spectral accuracy by the kernel truncation method. In addition, the influence of the model parameters on the eigenvalue distribution is studied and for various dipole orientations and an anisotropic external potential the phase diagrams of the eigenmodes are presented. Examples verify the spectral accuracy of the method.

AMS subject classifications: 35Q40, 35Q41, 65M70, 65T40, 65T50

Key words: Bogoliubov-de Gennes excitation, Bose-Einstein condensate, convolution-type nonlocal interaction, spectral method, kernel truncation method.

1. Introduction

The Bose-Einstein condensate (BEC), known as the "fifth state of matter", was theoretically predicted by Bose and Einstein at the beginning of the last century. Since 1995, the realisation of the BEC of dilute alkalis metal atoms opens up a new direction in the study of ultra-cold atoms [2]. Over the past few years, physicists have been looking for a novel type of quantum gases with dipolar interaction, acting between particles with permanent magnetic or electric dipole moments. It is possible to explore the dipolar BEC of ultracold atomic in experiments due to the remarkable discovery of ^{52}Cr atoms in 2005 [26]. A dipolar BEC with ^{164}Dy atoms, whose dipole-dipole interaction (DDI) is much stronger than that of ^{52}Cr, was achieved in experiments in 2011 [32]. In 2012, a new dipolar BEC of ^{168}Er atoms has been realised at the Insbruck University [1]. These experiments show that apart from early BECs, the DDI of dipolar BEC is anisotropic and long-range ones and

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this produces some unique phenomena. All of these have greatly promoted the theoretical and numerical investigations of dipolar BECs.

If the temperature T is much lower than a critical temperature T_c, the evolution of quasi-2D dipolar BEC is described by a macroscopic wave function $\psi = \psi(\mathbf{x},t)$, which satisfies the 2D Gross-Pitaevskii equation (GPE) with DDI term [7,9,43]

$$i\partial_t\psi(\mathbf{x},t) = \left[-\frac{1}{2}\Delta + V(\mathbf{x}) + \beta|\psi|^2 + \lambda\Phi(\mathbf{x},t)\right]\psi(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^2, \quad t>0, \tag{1.1}$$

$$\Phi(\mathbf{x},t) = \left(U * |\psi|^2\right) = \int_{\mathbb{R}^2} U(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}', \quad \mathbf{x}\in\mathbb{R}^2, \quad t\geq 0 \tag{1.2}$$

and initial data

$$\psi(\mathbf{x},t=0) = \psi_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^2. \tag{1.3}$$

Here, $\mathbf{x} = (x,y)^T \in \mathbb{R}^2, t$ is the time, $\rho(\mathbf{x},t) := |\psi(\mathbf{x},t)|^2$ the density, β a dimensionless interaction constant (positive for repulsive interaction and negative for attractive interaction), and $\Phi(\mathbf{x},t)$ is the real-valued nonlocal (long-range) DDI defined as the convolution of the interaction kernel $U(\mathbf{x})$ and a density function ρ. Besides, $V(\mathbf{x})$ is a given real-valued external trapping potential determined by the type of the system under consideration. In most of the BEC experiments, the harmonic potential $V(\mathbf{x})$ is chosen to trap the condensate — i.e.

$$V(\mathbf{x}) = \frac{1}{2}\left(\gamma_x^2 x^2 + \gamma_y^2 y^2\right),$$

where $\gamma_x > 0$ and $\gamma_y > 0$ are dimensionless constants proportional to the trapping frequencies in x- and y-directions, respectively. Moreover, λ is a constant characterising the strength of DDI and $U(\mathbf{x})$ is a long-range DDI potential. Here, $U(\mathbf{x})$ has the form

$$U(\mathbf{x}) = -\frac{3}{2}\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\nabla_\perp^2\right)\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^2}\frac{e^{-s^2/2}}{\sqrt{|\mathbf{x}|^2+\epsilon^2 s^2}}\mathrm{d}s, \quad \mathbf{x}\in\mathbb{R}^2 \tag{1.4}$$

with a given unit vector $\mathbf{n} = (n_1,n_2,n_3)^T$, i.e. $\|\mathbf{n}\|_{l^2} = \sqrt{n_1^2+n_2^2+n_3^2} = 1$, representing the 3D dipole axis [7], and

$$\nabla_\perp = (\partial_x,\partial_y)^T, \quad \mathbf{n}_\perp = (n_1,n_2)^T, \quad \partial_{\mathbf{n}_\perp} = \mathbf{n}_\perp\cdot\nabla_\perp, \quad \partial_{\mathbf{n}_\perp\mathbf{n}_\perp} = \partial_{\mathbf{n}_\perp}(\partial_{\mathbf{n}_\perp}).$$

As the confinement gets stronger and by a formal analysis — c.f. Refs. [9,21], we have

$$U \to U^\infty(\mathbf{x}) := -\frac{3}{2}\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\nabla_\perp^2\right)\frac{1}{2\pi}\frac{1}{|\mathbf{x}|}, \quad \text{as} \quad \epsilon\to 0. \tag{1.5}$$

The quasi-2D GPE (1.1)-(1.3) conserves two important quantities — viz. the total mass (or normalisation) of the wave function

$$N\big(\psi(\mathbf{x},t)\big) := \|\psi(\mathbf{x},t)\|^2 = \int_{\mathbb{R}^2}|\psi(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \equiv N\big(\psi(\mathbf{x},0)\big)$$