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Analytic Riemann Theta Function Solutions of Coupled Korteweg-de Vries Hierarchy

Minxin Jia, Xianguo Geng*, Yunyun Zhai, Jiao Wei and Huan Liu

School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, P.R. China.

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Abstract. Coupled Korteweg-de Vries hierarchy associated with a 3×3 matrix spectral problem is derived via a stationary zero-curvature equation and Lenard recursion equations. Resorting to the characteristic polynomial of the Lax matrix for coupled Korteweg-de Vries hierarchy, we introduce a trigonal curve $\mathscr{K}_g$ with three infinite points and establish the corresponding Baker-Akhiezer function and a meromorphic function on $\mathscr{K}_g$. Coupled Korteweg-de Vries equations are decomposed into systems of ordinary differential equations of Dubrovin-type. Analytic Riemann theta function solutions are obtained by using asymptotic expansions of the Baker-Akhiezer function and a meromorphic function and their Riemann theta function representations.

AMS subject classifications: 35Q51, 37K10, 35Q58, 35L65

Key words: Coupled KdV hierarchy, trigonal curve, Riemann theta function solution.

1. Introduction

The study of analytic solutions of the soliton equations is an important area of modern mathematics, which has numerous applications in physics and other sciences. In particular, analytic solutions such as solitons, breather solutions, rogue wave solutions are obtained in [14, 17, 28, 29] by the inverse scattering transformation, by the Darboux transformation, and by an algebro-geometric method — cf. [1, 18–20, 35] and references therein.

Riemann theta function solutions appear in the description of the quasi-periodic behaviour of nonlinear phenomena or the integrability of soliton equations. Besides, they are also used in order to find multi-soliton solutions, elliptic function solutions, and others [2, 4, 35]. Over the past decades, various efficient algorithms have been developed to construct the solutions mentioned for the soliton equations associated with 2×2 matrix spectral problems such as the Korteweg-de Vries (KdV), the AKNS, the Toda lattice, the Ablowitz-Ladik equations [6, 10, 15, 16, 26, 32, 38, 39]. On the other hand, it is difficult to construct Riemann theta function solutions of the soliton equations associated 3×3 matrix

*Corresponding author. *Email address:* `xggeng@zzu.edu.cn` (X. Geng)

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spectral problems, since they are connected to trigonal curves rather than to hyperelliptic curves as in 2×2 matrix spectral problems [3,5,11,36]. Nevertheless, Dickson *et al.* [8,9] proposed a unified approach to Riemann theta function solutions of the entire Boussinesq hierarchy related to a third-order differential operator. Geng *et al.* [21,22,24,40] further developed the method to deal with the soliton equations associated with 3×3 matrix spectral problems, such as the modified Boussinesq, the Kaup-Kupershmidt, the coupled modified KdV hierarchies, and so on.

The aim of this work is to construct explicit Riemann theta function solutions of coupled KdV hierarchy associated with a 3×3 matrix spectral problem and trigonal curves. Coupled KdV equations find various applications in atmospheric dynamical system, shallow stratified liquid [12,13], etc. Thus Lou *et al.* [31] derived new types of coupled KdV systems

$$\begin{aligned} v_t &= \alpha_1 v_{xxx} + \alpha_2 v v_x + \alpha_3 (vw)_x + \alpha_4 w w_x + \alpha_5 w v_x, \\ w_t &= \delta_1 w_{xxx} + \delta_2 w w_x + \delta_3 (vw)_x + \delta_4 v v_x + \delta_5 w v_x, \end{aligned} \tag{1.1}$$

which is used to describe atmospheric and oceanic phenomena such as atmospheric blocking, interactions of atmosphere and ocean [37], oceanic circulations and even hurricanes or typhoons [30]. If $\alpha_1 = \delta_1 = 1, \alpha_2 = \delta_2 = -6, \alpha_3 = \delta_3 = -3, \alpha_4 = \alpha_5 = \delta_4 = \delta_5 = 0$, the Eqs. (1.1) take the form

$$\begin{aligned} v_t &= v_{xxx} - 6 v v_x - 3(vw)_x, \\ w_t &= w_{xxx} - 6 w w_x - 3(vw)_x, \end{aligned} \tag{1.2}$$

which represents a reduction of the second nontrivial member in coupled KdV hierarchy.

This paper is organised as follows. In Section 2, we introduce a 3×3 matrix spectral problem with three potentials and derive coupled KdV hierarchy based on Lenard recursion equations and a zero-curvature equation. In Section 3, we introduce a trigonal curve, the Baker-Akhiezer function, and the corresponding meromorphic functions on the curve. In addition, we decompose coupled KdV hierarchy into an ordinary differential system. The asymptotic behaviour and the divisors of a meromorphic function and the Baker-Akhiezer function are studied in Section 4. In the last section we use three kinds of Abelian differentials in order to construct the Riemann theta function solutions of the coupled KdV hierarchy.

2. Coupled KdV Hierarchy

Here we are going to derive the coupled KdV hierarchy associated with a 3×3 matrix spectral problem

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} \lambda + u & v & w \\ 1 & -\frac{1}{2}u & 0 \\ 1 & 0 & -\frac{1}{2}u \end{pmatrix}, \tag{2.1}$$

where u, v, w are potentials and λ is a constant spectral parameter. To this end, we first solve the stationary zero-curvature equation