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An Efficient Feature-Preserving Image Denoising Algorithm Based on a Spatial-Fractional Anisotropic Diffusion Equation

Maoyuan Xu and Xiaoping Xie*

School of Mathematics, Sichuan University, Chengdu 610065, China.

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Abstract. An efficient feature-preserving fractional image denoising algorithm based on a nonlinear spatial-fractional anisotropic diffusion equation is proposed. Two-sided Grünwald-Letnikov fractional derivatives used in the PDE model are suitable to depict the local self-similarity of images. The short memory principle is employed to simplify the approximation scheme. Experimental results show that the method has an extremely high structural retention property and keeps a remarkable balance between noise removal and feature preserving.

AMS subject classifications: 65M10, 78A48

Key words: Image denoising, feature preserving, spatial-fractional diffusion equation, two-sided derivative, Grünwald-Letnikov derivative.

1. Introduction

With the development of modern society, the form and quantity of information grow rapidly. Images play an important role in this process but during acquisition, transmission and storage, they inevitably suffer from noise interference. Such interference can blur the images and lead to problems in the identification of the required targets, thus seriously affecting their subsequent processing and analysis. Therefore, the image denoising is a particularly important area in the image processing, and its challenging aspect is how to efficiently remove the noise while preserving the image structure features. The list of various algorithms developed for the image denoising includes convolutional filter based methods [15, 28], wavelet analysis based methods [8, 30], machine learning based methods [37, 41], and variation or PDE based methods. However, we note that although a conventional convolutional filter based method is easy to implement, it often has an ordinary performance. On the other hand, a wavelet based method demonstrates a better performance with an appropriate chosen threshold, but the Gibbs phenomenon appears at

*Corresponding author. *Email addresses:* 663521583@qq.com (M. Xu), xpxie@scu.edu.cn (X. Xie)

discontinuous points of the signal. In machine learning based method, high quality training samples are needed, so that the construction of an appropriate network architecture requires additional efforts.

Compared to other methods, the variation or partial differential equation (PDE) related methods are based on a rigorous mathematics and are easier to present, but more efforts should be spent on the construction of efficient method related formulas. In variation based methods, a given energy functional is used to establish an optimisation problem, the numerical solution of which will allow to restore the image in question — cf. [6, 7, 9, 16, 19, 27, 29, 32–34, 36].

The PDE based image denoising methods employ PDE directly derived from the mathematical properties of the diffusion and the structural characteristics of the image. Thus in order to encourage intra-region smoothing instead of inter-region smoothing, Perona and Malik [25] introduced an algorithm, later called the PM model, which used a diffusion process with a spatially varied diffusion coefficient. Experimental results show that this method provides remarkable edge preserving. However, the method is based on a second-order PDE and processed images suffer from the staircase effect (the piecewise constant effect) which makes the image artificial. It is also worth noting that many other methods based on second-order PDEs have the same drawback — cf. [4, 17, 31, 33, 35]. In order to solve this problem, fourth-order PDE based methods have been studied [5, 12, 18, 38]. Yet these methods smooth the image too much and the speckle effect appears.

In recent years, fractional PDE based methods have attracted more attention because they can balance noise removal and preserve image edges and textures [1, 10, 11, 20, 22, 39, 42]. Numerical experiments reveal that these methods are able to alleviate both the staircase and speckle effects and generate processed images of higher quality than the integer order PDE based methods. We also note fractional derivatives are also used in variation based methods — cf. [2, 3, 13, 14, 21, 23, 27, 29, 40]. However, so far there are only a few fractional PDE methods, which exploit spatial fractional derivatives. In particular, Liao [22] applied the spatial fractional order anisotropic diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}^{\alpha}\left[c\left(\|G_{\sigma_1} \cdot \nabla^{\alpha} u\|\right)\nabla^{\alpha} u\right]$$

to low-dose computed tomography imaging, with the Grünwald-Letnikov fractional derivative used in the definition of the fractional divergence and gradient operators $\operatorname{div}^{\alpha}$, ∇^{α}, $\alpha \in [0.5, 1.5]$. Zhang *et al.* [42] employed the spatial fractional telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t} - \operatorname{div}^{\alpha}\left(g\left(|\nabla^{\alpha} G_{\sigma} * u|\right)\nabla^{\alpha} u\right) = 0$$

in image structure preserving denoising, with the Riemann-Liouville fractional derivatives defining $\operatorname{div}^{\alpha}$, ∇^{α}, $\alpha \in [1, 2]$. Following the Rudin-Osher-Fatemi model (ROF or TV) [32], Abirami *et al.* [1] considered image denoising with the spatial fractional isotropic diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}^{\alpha} u$$