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Improved Inexact Alternating Direction Methods for a Class of Nonlinear Complementarity Problems

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Abstract. Three types of improved inexact alternating direction methods for solving nonlinear complementarity problems with positive definite matrices and nonlinear terms are proposed. The convergence of the methods is proven. Numerical examples confirm the theoretical analysis and show that the methods have advantages over similar existing methods, especially in large size problems.

AMS subject classifications: 90C33, 65F10, 65F50, 65G40

Key words: Inexact alternating direction method, nonlinear complementarity problem, successive overrelaxation, iterative method, symmetric positive definite.

1. Introduction

Complementarity problems (CPs), first introduced by Cottle, are widely used in computing the equilibrium points of bimatrix games [11, 28]. Such problems arise in scientific computing, engineering and economy when considering numerous problems in elasticity, traffic network design, optimal control, free boundary problems of fluid dynamics, asset pricing and image processing [14, 19, 20, 23, 37, 60].

Over the past several decades, various numerical algorithms have been proposed to solve nonlinear complementarity problems (NCPs). On the other hand, Mangasarian [34] showed the equivalence of CPs to systems of nonlinear equations and Mangasarian and Solodov [35] reformulated NCPs as unconstrained and constrained minimisation problems, Noor [38] employed several fixed point methods to complementarity problems. Other approaches include projection-type methods [47], merit functions [16], smooth and nonsmooth Newton methods [10, 13, 27, 40, 41], interior-point methods [50], linearisation methods [18], domain decomposition methods [1, 26]. Existence and uniqueness results for complementarity problems are summarised in surveys and books [17–19, 23]. Using the solvers based on matrix splitting and multisplitting, Bai [2, 3] and Bai and Wang [9]

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developed parallel nonlinear monotonically convergent multisplitting relaxation methods. Asynchronous parallel nonlinear multisplitting relaxation methods for large sparse NCPs have been studied in [5, 6]. These methods are very efficient on high-speed MIMD multi-processor systems since they allow to avoid the synchronous delays between corresponding processors. Matrix splitting methods are special cases of these multisplitting methods.

Here, we focus on a class of weakly NCPs. Recall that Bai [4] first considered the systems of weakly nonlinear equations, also called the mildly nonlinear systems [8]. Linear complementarity problems (LCPs) represent a special case of NCPs and in recent years, a number of LCP solvers — viz. modulus-based matrix splitting (MMS) iteration methods, attracted considerable attention [7, 31, 39], since very often they perform better than projected relaxation methods [12] and modified modulus methods [15]. In particular, Mezzadri [36] and Li [32] used MMSs to solve horizontal linear complementarity problems and second-order cone LCPs, respectively. MMS iteration methods have been also applied to a class of weakly NCPs — cf. [25, 45]. It turns out that MMSs are more efficient than the fixed point method [38] and the Fischer-Burmeister semismooth Newton method [13]. Zheng [53] improved the convergence of MMS iteration method for NCPs with H-matrices. The convergence results have been then extended from H-compatible splitting to H-splitting of NCPs with H_+-matrices [30]. Extensions of MMS methods, such as modified MMS method [33], accelerated MMS method [29], relaxed MMS method [58], two-step MMS method [46, 55], preconditioned MMS method [54, 59] and modulus-based matrix multisplitting methods [43,44,48,49], developed in recent years improved the original one. Besides, a modulus-based nonsmooth Newton method and sign-based methods are studied in [57] and [56], respectively.

In recent years, alternating direction methods (ADMs) have been applied to LCPs of the free boundary problems of mathematical physics [21, 22, 42]. Zhang *et al.* [51, 52] developed inexact ADMs of multipliers for solving a kind of LCPs. These methods were applied to weakly NCPs in He *et al.* [24]. The main contribution of our work is as follows:

- Three new numerical methods for solving weakly NCPs are proposed and their convergence is analysed.
- The methods have their own advantages and are much more efficient than the existing methods.

In practical applications, the system matrix A of weakly NCPs is usually large, sparse, symmetric and positive definite. This motivates the current study on different inexact ADMs. In Section 2, a direct inexact ADM, a symmetric successive overrelaxation (SSOR) based inexact ADM and a modified SSOR-based inexact ADM are introduced. Convergence of these methods is studied in Section 3. The numerical examples presented in Section 4, illustrate the theoretical results. Section 5 contains concluding remarks.

2. Improved Inexact ADMs

A nonlinear complementarity problem with respect to a vector-valued function $F : D \to \mathbb{R}^n$, where D is a subset of $\mathbb{R}^n$ containing at least the nonnegative cone, consists in finding