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A Support Vector Machine Method for Two Time-Scale Variable-Order Time-Fractional Diffusion Equations

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Abstract. An efficient least-square support vector machine (LS-SVM) method for a two time-scale variable-order time-fractional diffusion equation is developed. The method is particularly suitable for problems defined on complex physical domains or in high spatial dimensions. The problem is discretised by the L1 scheme and the Euler method. The temporal semi-discrete problem obtained is reformulated as a minimisation problem. The Karush-Kuhn-Tucker optimality condition is used to determine the minimiser of the optimisation problem and, hence, the solution sought. Numerical experiments show the efficiency and high accuracy of the method.

AMS subject classifications: 65M22, 65M70

Key words: Least-square support vector machine, variable-order, time-fractional diffusion equation, irregular domain.

1. Introduction

The classical Fickian diffusion partial differential equation (PDE) was derived under the assumptions that the underlying particle jumps have a mean waiting time and a finite variance [30]. The latter is characterised by solutions with Gaussian type symmetric and exponentially decaying tails and have been observed for the diffusive transport of solute in homogeneous porous media [2] under certain assumptions. However, the transport of solute in heterogeneous porous media exhibits power-law decaying tails, which probably do not accurately modelled by the Fickian diffusion PDE [30]. The time-fractional diffusion equation (tFDE) is derived via a continuous time random walk under the assumption that

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the corresponding waiting time probability density function has a power-law decaying tail — cf. [27,30],

$$\partial_t^\alpha u - \Delta u = f(\boldsymbol{x},t), \quad 0<\alpha<1, \tag{1.1}$$

where ∂_t^α is the Caputo fractional differential operator defined by

$${}_0I_t^\alpha g(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s)}{(t-s)^{1-\alpha}}ds, \quad \partial_t^\alpha g(t) := {}_0I_t^{1-\alpha} g'(t)$$

and $\Gamma(\alpha)$ refers to the Gamma function [31]. It can accurately describe the power-law decaying behavior of the subdiffusive transport of solute in heterogeneous media. Furthermore, in such applications as bioclogging [1], nonconventional hydrocarbon or gas recovery [11], design of shape memory polymers [20], manufacturing of viscoelastic materials [31] and biomaterials in orthopedic implants [42], the structure of porous materials may evolve in time. Since the order α of the tFDE (1.1) is related to the fractal dimension of the porous material via the Hurst index [27], these problems lead to variable-order tFDEs [7,24,34,38,44,47].

As is recently shown [36,41,46], the two time-scale variable-order tFDEs

$$\begin{aligned} &\partial_t u + k(t)\partial_t^{\alpha(t)} u - \Delta u = f(\boldsymbol{x},t), && (\boldsymbol{x},t)\in\Omega\times(0,T], \\ &u(\boldsymbol{x},0)=u_0(\boldsymbol{x}), && \boldsymbol{x}\in\Omega, \\ &u(\boldsymbol{x},t)=g(\boldsymbol{x},t), && (\boldsymbol{x},t)\in\partial\Omega\times[0,T] \end{aligned} \tag{1.2}$$

retain the long-term subdiffusive behavior of the typical tFDE (1.1). Moreover they are able to eliminate the nonphysical initial weak singularity behavior of the classical constant-order tFDE (1.1) [8,12,22,23,32,33,37], which can properly address the impact of the deformed porous media. Here, $\Omega\subset\mathbb{R}^d$ is a bounded domain with the boundary $\partial\Omega$, $\boldsymbol{x}:=(x_1,\dots,x_d)$, f, u_0, and g are prescribed source term, initial data, and boundary data, respectively. According to [24,34,38], the variable-order fractional differential operator $\partial_t^{\alpha(t)}u$ is defined by

$$\partial_t^{\alpha(t)} u(\boldsymbol{x},t) := \frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{\partial_s u(\boldsymbol{x},s)}{(t-s)^{\alpha(t)}}ds, \quad 0\le\alpha(t)\le\alpha_*<1. \tag{1.3}$$

In this paper we develop a least-square support vector machine (LS-SVM) method for the two time-scale variable-order tFDE (1.2) with a fast solution technique by exploring the feature of the governing equation. Our goal is the simulation of problems on complex physical domains or in high spatial dimensions. The rest of the paper is organised as follows. In Section 2 we reformulate the variable-order tFDE (1.2) as a minimisation problem and obtain an LS-SVM spatial discretisation by enforcing the variable-order tFDE (1.2) at a series of collocation points. In Section 3, we discretise the considered problem in time using the $L1$ scheme and the Euler method, derive the Karush-Kuhn-Tucker (KKT) optimality condition and thus the numerical solution of the variable-order tFDE (1.2). Section 4 presents a recently developed technique to improve the efficiency of the LS-SVM method. In Section 5 we carry out numerical experiments to investigate the performance of the developed LS-SVM scheme. Section 6 contains concluding remarks.