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On Parameterised Quadratic Inverse Eigenvalue Problem

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Abstract. It is shown that if prescribed eigenvalues are distinct, then the parameterised quadratic inverse eigenvalue problem is equivalent to a multiparameter eigenvalue problem. Moreover, a sufficient condition for the problem solvability is established. In order to find approximate solution of this problem, we employ the Newton method based on the smooth *QR*-decomposition with column pivoting and prove its locally quadratic convergence. Numerical examples illustrate the effectiveness of the method.

AMS subject classifications: 65F15

Key words: Quadratic inverse eigenvalue problem, multiparameter eigenvalue problem, smooth *QR*-decomposition, Newton method.

1. Introduction

There is a wide class of problems, which can be reduced to the system of second-order differential equations

$$M\ddot{x}(t)+C\dot{x}(t)+Kx(t)=f(t), \tag{1.1}$$

where $M,C,K\in\mathbb{R}^{n\times n}$, $\det M\neq 0$ are respectively the analytical mass, damping, and stiffness matrices and $x(t),f(t)\in\mathbb{R}^n$ the displacement and external force vectors depending on time t. Physical and geometric parameters characterising the underlying physical system are embedded in the coefficient matrices M,C and K. From a priori known physical and geometric parameters such as mass, damping coefficient, elasticity, length, area and so on, the process of analysing and deriving the dynamical behavior of the system is referred to as the direct problem. It is well known that if $x(t)=e^{\lambda t}v$, $\lambda\in\mathbb{C}$, $v\in\mathbb{C}^n$ a fundamental solution to (1.1), then λ and v satisfy the quadratic eigenvalue problem (QEP)

$$Q(\lambda)v=0, \tag{1.2}$$

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where $Q(\lambda) = \lambda^2 M + \lambda C + K$ and is called the quadratic pencil. Since M is a nonsingular matrix, the dynamical behavior of the system (1.1) may be interpreted via $2n$ eigenvalues $\{\lambda_i\}_{i=1}^{2n}$ and eigenvectors $\{v_i\}_{i=1}^{2n}$ of the QEP (1.2). Various quadratic eigenvalue problems have been studied so far and we refer the readers to [31] for a survey on their applications, mathematical properties, and corresponding numerical methods.

On the other hand, the inverse problem is to determine physical and geometric parameters for three matrices M, C, K such that the corresponding system has a prescribed dynamical behavior. More exactly, we consider the following problem.

Problem 1.1 (Parameterised Quadratic Inverse Eigenvalue Problem (PQIEP)). *Let $M \in \mathbb{R}^{n\times n}$ be a non-singular matrix and C_i, $K_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \ldots, 2n$. For any distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$, find a vector $c = (c_1, c_2, \ldots, c_{2n})^T \in \mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$ such that the quadratic eigenvalue problem*

$$\left[\lambda^2 M + \lambda C(c) + K(c)\right] x = 0 \tag{1.3}$$

with the matrices

$$C(c) := C_0 + \sum_{i=1}^{2n} c_i C_i, \ K(c) := K_0 + \sum_{i=1}^{2n} c_i K_i \tag{1.4}$$

has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$.

Such problems appear in various practical applications including structural design [22] and finite element model updating [7, 8]. The general theory, numerical methods and applications of the standard inverse eigenvalue problem — i.e. if $M = 0$ and $C(c)$ is the identity matrix, are discussed in [6, 18, 33, 34]. Mathematical theory and algorithms of the parameterised generalized inverse eigenvalue problem — i.e. if $M = 0$, are studied in [9, 11, 12, 22, 29].

Using the determinant evaluations proposed by Lancaster [24] and Biegler-König [3] and further developed by Friedland *et al.* [18], Elhay and Ram [16] reformulated PQIEP as the system of nonlinear equations

$$F(c) = \begin{pmatrix} \det\left(\lambda_1^2 M + \lambda_1 C(c) + K(c)\right) \\ \det\left(\lambda_2^2 M + \lambda_2 C(c) + K(c)\right) \\ \vdots \\ \det\left(\lambda_{2n}^2 M + \lambda_{2n} C(c) + K(c)\right) \end{pmatrix},$$

and proposed a Newton method for its solution. However, this method may suffer from ill-conditioning [18] and is not computationally attractive.

Assume that Problem 1.1 has a solution c^*. Then there is a neighborhood of c^* such that the eigenvalues $\lambda_i(c)$ of the quadratic pencil $Q_c(\lambda) = \lambda^2 M + \lambda C(c) + K(c)$ are distinct differentiable functions [1]. In this neighborhood, Elhay and Ram [17] considered the