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Numerical Simulations of Stochastic Differential Equations with Multiple Conserved Quantities by Conservative Methods

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Abstract. The deterministic discrete gradient method for stochastic differential equations is extended to equations with multiple conserved quantities. The equations with multiple conserved quantities in the Stratonovich sense are written in the skew-gradient form, which is used in the construction of the stochastic discrete gradient method. It is shown that the stochastic discrete gradient method has the mean-square convergence order one and preserves all conserved quantities. Besides, for a given skew-gradient form, the stochastic discrete gradient method is equivalent to the stochastic projection method. Numerical examples confirm the theoretical results and show the effectiveness of the method.

AMS subject classifications: 65M10, 78A48

Key words: Stochastic differential equation, multiple conserved quantity, discrete gradient, projection, mean-square convergence.

1. Introduction

Stochastic differential equations (SDEs) are an important tool for describing stochastic phenomena arising in physics, engineering, economics, chemistry, and biology [7,12,17,18, 20]. However, analytical solutions of such equations are rarely available. Therefore, there are various numerical methods developed for their solution — cf. [2,3,11,13,25,28–30]. In the last decades, the numerical methods preserving the geometric invariants along the flows, such as symplectic and Lie group structures, and quantities showing that the exact solution evolves on a manifold of the dimension smaller than that of $\mathbb{R}^n$, have been widely studied [5,8,9,27,32]. Such numerical approaches are called the structure-preserving or geometric numerical integration (GNI) methods. They have a number of advantages and provide reliable numerical solutions. Therefore, GNI for SDEs with geometric attributes

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attracted a considerable attention [4,14,19,24,33]. In this work, we are mainly interested in SDEs with the conserved energy and momentum.

Consider the following general autonomous SDEs:

$$\mathbf{d}x(t)=\sum_{i=0}^{d}G^i\big(x(t)\big)\circ\mathbf{d}W^i(t),\quad x(0)=x_0, \tag{1.1}$$

where "$\circ$" refers to the Stratonovich integral, $W^0(t):=t$ and $W^i(t)$, $i=1,\dots,d$ are independent one-dimensional standard Wiener processes defined on a complete filtered probability space $(\Omega,\mathscr{F},\mathbb{P},\{\mathscr{F}_t\}_{t\in[0,T]})$ and satisfying the usual conditions. Besides, the initial value x_0 is a $\mathbb{R}^n$-valued random variable such that $E|x_0|^2<\infty$, $|\cdot|$ the Euclidean norm, and the functions $G^i:\mathbb{R}^n\to\mathbb{R}^n$, $i=0,\dots,d$, satisfy the conditions under which (1.1) has a unique solution [20].

To simplify the presentation, throughout this paper we use the Einstein's summation convention — i.e. repeated indices in a single term mean the summation of such terms over all range of the indices. Thus the Eq. (1.1) can be written as

$$\mathbf{d}x(t)=G^i\big(x(t)\big)\circ\mathbf{d}W^i(t). \tag{1.2}$$

Assume that the Morse functions $I^m:\mathbb{R}^n\to\mathbb{R}$, $m=1,\dots,M$ are conserved quantities of (1.2), i.e.

$$(G^i)^\top\nabla I^j=0,\quad i=0,\dots,d,\quad j=1,\dots,M,$$

where ∇I^j denotes the gradient of I^j. This means that the exact solution evolves on the manifold

$$\mathscr{M}_{x_0}=\Big\{y\in\mathbb{R}^n|I^m(y)=I^m(x_0)\ \text{ for }\ m=1,\dots M\Big\}.$$

When constructing numerical methods for SDEs with multiple conserved quantities, it is natural to require the numerical solutions to preserve all multiple conserved quantities. We note that there are two different sets of numerical methods which preserve conserved quantities of SDEs — viz. the ones that use the terms I^m explicitly and the ones that do not. The latter include symplectic methods automatically preserving the quadratic Hamiltonian energy [24] and the methods preserving the matrix Lie group structure, which are equivalent to the preservation of conserved quantity determined by the corresponding Lie group [1,19]. The former comprise stochastic averaged vector field methods [6], stochastic projection methods [15,34] and stochastic discrete gradient methods [10,15].

It is worth noting that most of the existing works are devoted to SDEs with single conserved quantity. On the other hand, to the best of our knowledge there are only a few researches concerning conservative numerical methods for SDEs with multiple conserved quantities. Thus Chen *et al.* [4] modified the stochastic averaging vector field (AVF) method to preserve multiple conserved quantities, Zhou *et al.* [34] introduced multiple Lagrange multipliers and directly applied a linear projection method to SDEs with multiple conserved quantities. In comparison to the case of a single conserved quantity, this can increase the computational costs. In this paper, we extend a stochastic gradient method for SDEs with single conserved quantity [10] to SDEs with multiple conserved quantities and show that