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Time Domain Linear Sampling Method for Inverse Scattering Problems with Cracks

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Abstract. The paper analyses time-dependent scattering and inverse scattering problems with cracks. The well-posedness of the forward scattering problem is proved and a modified retarded potential boundary integral equation method is utilised to solve the forward problem. Besides, the inverse scattering problem of finding the cracks by using measured scattered data is considered. A time domain linear sampling method for inverse problems is developed and the blow-up property is proved. The computation scheme is relatively simple and easy to implement. Numerical examples demonstrate the effectiveness of the methods.

AMS subject classifications: 35L20, 65M32

Key words: Time domain, inverse scattering problem, cracks, boundary integral equation, linear sampling method.

1. Introduction

Scattering and inverse scattering problems with acoustic waves are important subjects in mathematical physics. Usually, the problems can be split into two groups — viz. frequency domain and time domain problems. The frequency domain problems, dealing mainly with the scattering of time harmonic waves, are extensively studied in the past decades [9–11]. The time domain problems, on the other hand, concerns the scattering of time-dependent waves and the wave equation. They are related to time variables naturally arising in practical problems and attracted a growing attention in recent years. Thus [6] investigates the reconstruction of time-dependent point sources and [19] employs the Schrödinger equation with time-dependent method.

This paper considers the scattering of time-dependent acoustic waves on infinite cylinders with cracks in $\mathbb{R}^2$ as cross section. The forward scattering problems can be solved by the retarded potential boundary integral equation (RPBIE) method [2, 14, 17, 24], which is based on the classical boundary integral equations [18, 25, 26]. The inverse problem

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consists in finding cracks from measured scattered data. The corresponding frequency domain problems are studied by Newton type iteration methods [21,22] based on the domain derivative [3,28], factorisation method [27] and the linear sampling methods [4,20]. However, time domain methods are rarely used in inverse problems. In this paper, a linear sampling method (LSM) is applied to a time domain inverse scattering problem. The LSM is relatively fast since it requires neither the iteration nor the forward solver. Moreover, the LSM is easy to implement since no a priori information about the unknown targets is needed. Therefore, it is widely used in frequency domain problems [1, 4, 5] and time domain problems [7, 8, 15, 16].

Although time domain LSM has been often used in the reconstruction of bounded scatterers [8, 16], the analysis of the inverse scattering problem for cracks is not obvious. The basis of the time domain LSM is the blow-up behavior of the indicator function for points approaching the boundary of the obstacle from the inside. The boundary can be shown by plotting an indicator function over the sampling grid. However, the crack has no concepts of inside and outside. Therefore, in this paper, a new test function is proposed and a new proof of the blow-up property is provided.

The outlines of this paper are as follows. Forward and inverse scattering problems and relevant functional spaces are introduced in Section 2. In Section 3, an RPBIE method and the well-posedness of forward scattering problems are provided. In Section 4, we prove the blow-up property of the time domain LSM for an inverse problem. Section 5 is devoted to the numerical implementation of the RPBIE for the forward problem and the time domain LSM for the inverse problem.

2. Problem Setting

Let $\rho : [0,1] \to \mathbb{R}^2$ be an injective and piecewise smooth function such that $\Gamma = \{\rho(s) : s \in [0,1]\}$ is a nonintersecting oriented open arc without cusps in $\mathbb{R}^2$, and let Γ_+ and Γ_- denote the right and left sides of Γ, respectively.

We consider the scattering of time-dependent acoustic waves on Γ with Dirichlet boundary conditions. The incident wave is chosen as a cylindrical wave [24] defined by the time convolution

$$u^i(x,t;y) := G(x,t;y) * \lambda(t), \tag{2.1}$$

where

$$G(x,t;y) = \frac{H(t - c^{-1}|x-y|)}{2\pi\sqrt{t^2 - c^{-2}|x-y|^2}}$$

is the Green's function of the d'Alembert operator in $\mathbb{R}^2$, $\lambda(t)$ a signal function, c the sound speed in the homogeneous background medium, and H the Heaviside function. In this work, $\lambda(t)$ is causal — i.e. $\lambda(t) = 0$ for all $t < 0$.

We split the total wave field u^{tot} into the incident field u^i and the scattered field u. Then the scattered field u satisfies the equations

$$c^{-2}\partial_{tt}u - \Delta u = 0 \qquad \text{in } \Omega \times \mathbb{R}, \tag{2.2}$$