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On Convergence of the Partially Randomized Extended Kaczmarz Method

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Abstract. To complete the convergence theory of the partially randomized extended Kaczmarz method for solving large inconsistent systems of linear equations, we give its convergence theorem whether the coefficient matrix is of full rank or not, tall or flat. This convergence theorem also modifies the existing upper bound for the expected solution error of the partially randomized extended Kaczmarz method when the coefficient matrix is tall and of full column rank. Numerical experiments show that the partially randomized extended Kaczmarz method is convergent when the tall or flat coefficient matrix is rank deficient, and can also converge faster than the randomized extended Kaczmarz method.

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1. Introduction

The Kaczmarz method [13] is an effective row-action iteration method for solving large systems of linear equations of the form

$$Ax = b \tag{1.1}$$

with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

If $A^{(i_k)}$ refers to the i_k-th row of the coefficient matrix A and $b^{(i_k)}$ to the i_k-th entry of the vector b, then at the k-th iterate of the Kaczmarz method the current iteration vector x_k is projected onto the hyperplane $\{x\,|A^{(i_k)}x = b^{(i_k)}\}$, i.e.

$$x_{k+1} = x_k + \frac{b^{(i_k)} - A^{(i_k)}x_k}{\|A^{(i_k)}\|_2^2}\left(A^{(i_k)}\right)^T, \tag{1.2}$$

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where $\|\cdot\|_2$ and $(\cdot)^T$, respectively, represent the Euclidean norm and the transpose of the corresponding matrix or vector, and $i_k = (k \mod m) + 1$. In computerized tomography, the above method is also called the algebraic reconstruction technique [12]. The convergence of the Kaczmarz method has been studied in [1, 20, 22].

Instead of the sequential selection of the rows of the matrix A, Strohmer and Vershynin [21] proposed to choose them randomly, according to a probability proportional to the squared Euclidean norm of each row. The corresponding method is called the randomized Kaczmarz method. It is proven that for consistent linear systems, this randomized Kaczmarz method converges with an expected exponential convergence rate to the least-norm solution of the system and the convergence factor is only related to the scaled condition number — cf. [5, 15, 21] and the references therein. After the randomized Kaczmarz method, many works about the Kaczmarz method came forth. Needell [17] derived an upper bound for the expected solution error of the randomized Kaczmarz method for consistent linear systems corrupted by noise. The randomized block Kaczmarz methods are discussed in [16, 18]. Bai and Wu proposed the greedy randomized Kaczmarz method [3, 4] for consistent linear systems which can converge faster than the randomized Kaczmarz method. The probabilities of selecting the rows in the randomized Kaczmarz method and the greedy randomized Kaczmarz method are also applied to the coordinate descent method for selecting the columns in the coefficient matrix [6, 14].

In order to treat inconsistent linear systems, Zouzias and Freris [24] extended the randomized Kaczmarz method and introduced the randomized extended Kaczmarz (REK) method. The variants of the REK method and their convergence are discussed in [8, 9, 11, 19, 23]. Changing the random selection of the columns in the REK method to a non-random one and keeping the selection of the rows random, Bai and Wu [7] proposed the partially randomized extended Kaczmarz (PREK) method. However, the convergence of this method has been proven only for tall coefficient matrices of full column rank. Nevertheless, the PREK method can also converge if the coefficient matrix has no full column rank and even if it is flat. In this paper, we remove the full column-rank condition in the convergence analysis for the PREK method and present a convergence result for arbitrary coefficient matrices $A \in \mathbb{R}^{m\times n}$. The upper bound for the expected solution error of the PREK method we found, is smaller than the one in [7] when the coefficient matrix is tall and of full column rank. We also use numerical experiments to show that if the coefficient matrix is not of full column rank, the PREK method can also converge to the least-norm least-squares solution of linear system and outperforms the REK method.

The rest of this paper is organized as follows. Section 2 contains notations, concepts, basic results, and the description of the PREK method. In Section 3 we recall known convergence result of the PREK method for tall full column-rank coefficient matrices, and we complete the convergence theory of the PREK method for systems with different kinds of coefficient matrices. Numerical examples presented in Section 4 show the performance of the PREK method. In particular, we apply this method to inconsistent linear systems with tall and flat coefficient matrices $A \in \mathbb{R}^{m\times n}$, which do not have full column rank. Besides, we compare the efficacy of the PREK and REK methods. Our concluding remarks are given in Section 5.