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A Deep Learning Method for Elliptic Hemivariational Inequalities

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Abstract. Deep learning method for solving elliptic hemivariational inequalities is constructed. Using a variational formulation of the corresponding inequality, we reduce it to an unconstrained expectation minimization problem and solve the last one by a stochastic optimization algorithm. The method is applied to a frictional bilateral contact problem and to a frictionless normal compliance contact problem. Numerical experiments show that for fine meshes, the method approximates the solution with accuracy similar to the virtual element method. Besides, the use of local adaptive activation functions improves accuracy and has almost the same computational cost.

AMS subject classifications: 65K15, 68T07, 68U99

Key words: Deep learning, elliptic hemivariational inequality, contact problem, mesh-free method.

1. Introduction

With the advance of deep learning technique originated in computer science, considerable efforts are made to use this approach in other areas, including numerical methods for partial differential equations (PDEs). Neural network-based numerical methods appeared in 1990s [29] and have been significantly improved recently [7, 8, 10, 25, 40, 43, 44, 47]. In those methods, deep neural networks (DNNs) are used in order to parameterize the PDE solution by the parameters defined by an optimization problem related to the PDE under consideration. The key to the success of neural networks-based methods lies in the universal approximation property [2, 9, 18, 20, 42]. It is well known that deep neural network is a powerful tool for solving high-dimensional PDEs — cf. [12, 13, 26, 36]. We are sure that this is also a valuable strategy to attack low-dimensional complicated problems in science and engineering, which are expensive to solve by traditional numerical methods. Following these ideas, Huang *et al.* [21] proposed deep learning-based methods for variational inequalities. This approach was supported by other authors [30, 41]. How-

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ever, to the best of our knowledge, there are no similar works on hemivariational inequalities (HVIs). We recall that HVIs, introduced in 1983 by Panagiotopoulos [38] in connection with engineering applications, have been rigorously studied [14, 37, 39] and are now widely used in contact mechanics. In practice, the solutions to elliptic HVIs are only available numerically. There are various numerical methods to approximate the solutions of elliptic HVIs — cf. [11, 15, 17, 46]. However, the discretization often leads to non-convex and non-smooth optimization problems, the solution of which is challenging. The iterative convexification [35, 45] is a popular method for solving the non-convex optimization problems mentioned. It consists in construction and solution of a number of convex problems approaching the original non-convex problem. Many HVIs related contact problems such as the frictional bilateral contact problem, the frictionless normal compliance contact problem and the frictionless unilateral contact problem have been solved by the method mentioned [16].

In addition to the iterative convexification approach, one can use the proximal bundle method [34], the bundle Newton method [33], and the primal-dual active-set algorithm [28]. Recently, Feng *et al.* [11] used the double bundle method [24] to solve discrete non-convex and non-smooth problems arising in the discretization of HVIs. Nevertheless, the numerical methods mentioned are rather difficult to implement or they are computationally expensive when applied to the corresponding discrete problems.

Here, we consider a deep learning method for an elliptic HVI. The method is based on an equivalent variational formulation of the problem [14]. In particular, the solution space of the HVI is parameterized by DNNs and an approximation is found by minimizing an unconstrained expectation minimization problem. The latter can be solved by stochastic gradient descent methods [3]. The unconstrained expectation minimization problem is reformulated by using the variational principle for HVIs. Therefore, the resulting deep learning optimization problem has a clear physical meaning. As applications, we employ the deep learning method to a frictional bilateral contact problem and a frictionless contact problem with normal compliance. In addition, we use a fixed activation function and a local adaptive activation function [23] to solve HVIs in numerical simulation. Numerical experiments also show that the deep learning method has the same accuracy as traditional numerical methods on fine grids. Besides, the use of local adaptive activation functions gives a better accuracy than fixed activation functions, under comparable computational cost. Finally, it is worth noting that the method is suitable for engineering applications and can be easily programmed in Python.

The rest of this paper is organized as follows. In Section 2, an elliptic HVI and its applications in contact mechanics are introduced. Section 3 provides a detailed description of the deep learning method for HVIs. In Section 4, two numerical examples demonstrate the efficiency of the deep learning methods. Finally, we summarize our work with a short conclusion in Section 5.

2. Elliptic Hemivariational Inequality and Applications

Let X be a real Banach space with norm $\|\cdot\|_X$ and X^* the dual of X with norm $\|\cdot\|_{X^*}$.