

Optimal Convergence Rates in Time-Fractional Discretisations: the L1, $\overline{\text{L1}}$ and Alikhanov Schemes

Yongtao Zhou and Martin Stynes*

Applied and Computational Mathematics Division, Beijing Computational Science Research Center, Beijing 100193, China.

Received 29 June 2021; Accepted (in revised version) 22 September 2021.

Abstract. Consider the discretisation of the initial-value problem $D^\alpha u(t) = f(t)$ for $0 < t \leq T$ with $u(0) = u_0$, where $D^\alpha u(t)$ is a Caputo derivative of order $\alpha \in (0, 1)$, using the L1 scheme on a graded mesh with N points. It is well known that one can prove the maximum nodal error in the computed solution is at most $\mathcal{O}(N^{-\min\{r\alpha, 2-\alpha\}})$, where $r \geq 1$ is the mesh grading parameter ($r = 1$ generates a uniform mesh). Numerical experiments indicate that this error bound is sharp, but no proof of its sharpness has been given. In the present paper the sharpness of this bound is proved, and the sharpness of the analogous nodal error bounds for the $\overline{\text{L1}}$ and Alikhanov schemes will also be proved, using modifications of the L1 analysis.

AMS subject classifications: 65L05, 65L70

Key words: L1 scheme, $\overline{\text{L1}}$ scheme, Alikhanov scheme, optimal convergence rate.

1. Introduction

Physical models that use fractional time derivatives have attracted a lot of recent attention from numerical analysts; see the survey paper [5]. Many of these models include a Caputo time derivative of order $\alpha \in (0, 1)$, defined [3] for absolutely continuous functions $u : [0, T] \rightarrow \mathbb{R}$ by

$$D^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^t (t-s)^{-\alpha} u'(s) ds \quad \text{for } 0 < t \leq T. \quad (1.1)$$

To solve the Caputo initial value problem

$$D^\alpha u(t) = f(t) \quad \text{for } 0 < t \leq T \quad (1.2)$$

with $u(0)$ given and f smooth, a popular discretisation of $D^\alpha u(t)$ is the L1 difference scheme. It is well known [12, 13] that typical solutions of (1.2) have a weak singularity at

*Corresponding author. *Email addresses:* yongtaozhou@csrc.ac.cn, yongtaozh@126.com (Y. Zhou), m.stynes@csrc.ac.cn (M. Stynes)

the initial time $t = 0$, and to address this difficulty the L1 scheme is usually implemented on the graded mesh

$$t_0 := 0, \quad t_n := T(n/N)^r, \quad \tau_n := t_n - t_{n-1} \quad \text{for } n = 1, 2, \dots, N, \quad (1.3)$$

where N is any positive integer and the mesh grading parameter $r \geq 1$ is chosen by the user. This mesh is used in Sections 2-4. A consistency and stability argument [13] bounds the error in the computed solution $\{u_n\}_{n=0}^N$ by

$$\max_{n=0,1,\dots,N} |u(t_n) - u_n| \leq CN^{-\min\{r\alpha, 2-\alpha\}} \quad (1.4)$$

with a constant C that depends on the data of the problem. Many papers for time-fractional initial-boundary problems have used the L1 discretisation (see [11] for a survey) and obtained error estimates identical to (1.4), plus an additional term for the spatial discretisation error, and confirmed the sharpness of (1.4) by numerical experiments.

In the present paper we shall prove that the bound (1.4) is optimal — i.e. show it is also a lower bound for the error and give direct explanations of the origins of these two terms: $N^{-r\alpha}$ is caused by the weak singularity at $t = 0$, while $N^{-(2-\alpha)}$ is due to effects far from $t = 0$.

Note that $\mathcal{O}(N^{-r\alpha})$ and $\mathcal{O}(N^{-(2-\alpha)})$ upper bounds for the error follow from [7, Remark 5] if there one takes $\sigma = \alpha$ and $\sigma = 2 - \alpha$ respectively; our lower bounds for the error complement this result.

The rest of the paper is concerned with related results for other discretisations of fractional derivatives of order $\alpha \in (0, 1)$. While the overall aims (viz., prove the sharpness of the error bound and show the origins of the terms appearing in it) are the same for each scheme, the details of these analyses can vary considerably from the L1 analysis.

First, we consider an averaged variant of the L1 scheme which we write as $\overline{\text{L1}}$; similar methods have been studied in [4, 9, 10] (in [4] the scheme is denoted by L1⁺). Numerical results indicate that the computed solution $\{u_n\}_{n=0}^N$ satisfies the error bound

$$\max_{n=0,1,\dots,N} |u(t_n) - u_n| \leq CN^{-\min\{r\alpha, 2\}}. \quad (1.5)$$

We shall prove the optimality of (1.5) by first showing that for the typical solution $u(t) = t^\alpha$ the error is at least $\mathcal{O}(N^{-r\alpha})$ on our graded mesh, then for the smooth solution $u(t) = t^2$ (for which a uniform mesh is adequate) we prove that the error in the computed solution is at least $\mathcal{O}(N^{-2})$. A similar upper bound for the error when $u(t) = t^2$ is derived in [4, Remark 2].

Alikhanov [1] constructed the $L2-1_\sigma$ discretisation of the Caputo fractional derivative D^α using piecewise quadratic interpolation of u . When the solution $\{u_n\}$ to (1.2) is computed on the graded mesh (1.3) by this method, it is known [2, Lemma 7 and Theorem 1] that the error satisfies

$$\max_{n=0,1,\dots,N} |u(t_n) - u_n| \leq CN^{-\min\{r\alpha, 3-\alpha\}}. \quad (1.6)$$