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Global Existence and Asymptotic Behavior in a Predator-Prey-Mutualist Model with Prey-Taxis

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Abstract. This paper considers the global existence and boundedness of classical solutions to a predator-prey-mutualist model with prey-taxis. In addition, by constructing the Lyapunov functionals, we proved when $\alpha < (ac/b)\cdot r + a/b$, the positive equilibrium point is globally asymptotic stable; and when $\alpha \in ((ac/b)\cdot r + a/b, M_1)$, the semi-trivial equilibrium point is globally asymptotic stable. Finally, we give some numerical examples to validate our results.

AMS subject classifications: 35A01, 35B40, 35J57, 35Q92

Key words: Chemotaxis, predator-prey-mutualist, boundedness, stabilization.

1. Introduction

Prey-taxis, the biased random walks, means that the movement of predator towards the area with higher density of prey population, plays an important role in biological control and ecological balance such as the regulation of the prey population below an economic threshold or incipient outbreaks of the prey [11, 22, 36]. It is known that in addition to the random diffusion of predator and prey, the spatiotemporal variation of the predator velocity in spatial predator-prey models is affected by the prey gradient [1,15,16]. Karevia and Odell [15] introduced the following PDE prey-taxis model:

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot \big(u\rho(u,v)\nabla v\big) + G_1(u,v), \\ v_t &= D\Delta v + G_2(u,v), \end{aligned} \tag{1.1}$$

where u and v are respectively the densities of predator and prey, $-\nabla \cdot (u\rho(u,v)\nabla v)$ is the prey-taxis with a coefficient $\rho(u,v)$ which may depend on the predator or prey density, and D the prey diffusion rate. The functions G_1 and G_2 describe the interaction between

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predator and prey populations. The mathematical analysis of (1.1) and the variants thereof concentrates on the boundedness and blow-up of solutions. When

$$G_1(u,v) = ru\frac{v}{\lambda+v} - u\theta, \quad G_2(u,v) = -u\frac{v}{\lambda+v} + \mu v\left(1-\frac{v}{K}\right) \tag{1.2}$$

and $\rho(u,v) = \rho_1(u)$ depends only on u but is truncated at an $u_* > 0$, i.e. if $\rho_1(u_*) = 0$ for $u > u_*$ and $\rho_1(u) > 0$ for $0 \leq u < u_*$. Ainseba *et al.* [1] obtained the global weak solutions of (1.1) with (1.2) for $n \geq 1$ by using the Schauder fixed point theorem and duality technique. In the case $n \leq 3$, Tao [27] extended the work [1] to the global classical solutions with the help of L^p estimates and Schauder estimates, which depends on time. He and Zheng [12] improved the results of [27] by proving the uniform-in-time boundedness of the solutions. If $\rho(u,v) = \chi > 0$ is a constant, the existence of non-constant steady states of (1.1) with (1.2) was studied in [17,32], where the Hopf bifurcation theorem and index degree theory were used. When

$$G_1(u,v) = \gamma uF(v) - uh(u), \quad G_2(u,v) = -uF(v) + f(v)$$

and $\rho(u,v) = \chi\rho_2(u)$, $\rho_2(u) \leq u$, Wu *et al.* [33] consider functional forms of $F(v), h(u)$ and $f(v)$ and show that if χ is small, then the solutions are globally bounded. The asymptotic behavior of solutions was studied for particularized predator-prey interactions under special conditions. In the case $\rho(u,v) = \chi u$, Jin and Wang [14] established an entropy-like equality, a boundedness criterion, and used it to show that the intrinsic interaction between predators and preys suffices to prevent the population overgrowth even if the strong prey-taxis is present. Besides, they provided conditions for the global asymptotical stability and co-existence of the prey-only steady state and established convergence rates of the solutions to the steady states. If $\rho(u,v) = u, F(v) = v, h(u) = a_1, f(v) = a_2 v$ and Δu is replaced by $d_1\Delta u$, Xiang [35] noted that for any regular initial data, the system (1.1) admits a unique global smooth solution for arbitrary size of χ. For $a_2 \leq 0$, it is uniformly bounded in time. The reader can also consult Refs. [6,7,21,30,31] for more results.

The mutualism is the interaction of various species, which enhances the per capita growth rate of the others. It occurs in many important processes and systems, including rhinoceros and Rhino birds, mycorrhizal associations, nitrogen, and perhaps even cell organelles [29]. Rai *et al.* [23] proposed and studied a general predator-prey-mutualist ODE model. Taking into account the diffusion mechanism and using spectral properties of linearized operators, Zheng [27] demonstrated that the boundary conditions can affect the asymptotic behaviour of the population. Tian *et al.* [29] constructed a PDE model for shrub ecosystem with mutualistic interaction and showed the spatial isolation of the shrub ecosystem. Li *et al.* [18] proved the uniqueness and non-uniqueness of steady states for a diffusive predator-prey-mutualist model with a protection zone (see also [13, 19, 36]). Guo and Wu [10] studied a diffusive predator-prey-mutualist model in a bounded domain $\Omega \subset \mathbb{R}^N$ with the smooth boundary $\partial\Omega$, viz.

$$\frac{\partial u}{\partial t} = d_u\Delta u + u\left(a - u - \frac{\alpha v}{1+rw}\right), \quad t > 0, \quad x \in \Omega, \tag{1.3a}$$