East Asian Journal on Applied Mathematics Vol. **12**, No. 3, pp. 590-598
doi: 10.4208/eajam.030921.141121 August 2022

Achieving Superconvergence by One-Dimensional Discontinuous Finite Elements: Weak Galerkin Method

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Received 3 September 2021; Accepted (in revised version) 14 November 2021.

Abstract. A simple stabilizer free weak Galerkin (SFWG) finite element method for a one-dimensional second order elliptic problem is introduced. In this method, the weak function is formed by a discontinuous k-th order polynomial with additional unknowns defined on vertex points, whereas its weak derivative is approximated by a polynomial of degree $k+1$. The superconvergence of order two for the SFWG finite element solution is established. It is shown that the elementwise lifted P_{k+2} solution of the P_k SFWG one converges at the optimal order. Numerical results confirm the theory.

AMS subject classifications: 65N15, 65N30

Key words: Finite element, weak Galerkin, stabilizer free.

1. Introduction

Weak Galerkin (WG) finite element methods introduced and analyzed in [2,3], provide a general finite element technique for solving partial differential equations. The novelty of such methods consists in using weak functions and their weakly defined derivatives. Weak functions have the form $v = \{v_0, v_b\}$ with $v = v_0$ representing v in the interior of each element and $v = v_b$ on the element boundary. The terms v_0 and v_b are respectively approximated by polynomials $P_k(T)$ and $P_s(e)$, where e refers to the edge or face of T. Weak derivative is specifically developed for weak function approximated by $P_\ell(T)$ polynomials. Each combination of the WG elements $(P_k(T), P_s(e), [P_\ell(T)]^d)$ leads to a weak Galerkin finite element method.

For special combinations of the WG elements $(P_k(T), P_s(e), [P_\ell(T)]^d)$, $d \geq 2$, the corresponding WG method does not need a stabilizer. This leads to stabilizer free weak Galerkin (SFWG) methods. The first stabilizer free weak Galerkin method was introduced

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in [4] on polygonal and polyhedral meshes. It was shown that if the corresponding polygon has n sides, then one can eliminate the stabilizer term by using the WG elements $(P_k(T), P_k(e), [P_{k+n-1}(T)]^d)$. Al-Taweel and Wang [1] improved this result by reducing the polynomial degree for weak gradient on triangular meshes. Besides, the superconvergence is observed for special WG elements. In particular, order one superconvergence of an SFWG method is established for the WG elements $(P_k(T), P_k(e), MRT_k(T))$, where $MRT_k(T)$ is a macro-Raviart-Thomas element on a polygon/polyhedron T [5]. Order two superconvergence is obtained for the WG elements $(P_k(T), P_{k+1}(e), [MP_{k+1}(T)]^d)$, where $[MP_{k+1}(T)]^d$ is a macro-BDM element on a polygon/polyhedron T [6].

We investigate the performance of the SFWG elements $(P_k(T), P_s(e), [P_\ell(T)]^d)$ in one dimension. In this case, the polynomial $P_s(e)$ degenerates to a single value at the end points of the interval. This work answers the question which WG elements $(P_k(I), P_0(x_i), P_\ell(I))$ maximize the order of convergence in one dimension, where x_i is the end point of the interval I. More exactly, we show that in the one dimensional case, one can obtain two order higher convergence rate for the solution of the SFWG method with the WG element $(P_k(T), P_0(x_i), [P_{k+1}(T)]^d)$, i.e. the P_k SFWG solution converges with order $k+3$ in L^2 norm and with order $k+2$ in H^1 norm. Moreover, the P_k SFWG solution is lifted to a P_{k+2} solution elementwise, which converges with the optimal order. Numerical results confirms the theoretical findings.

2. SFWG Finite Element Schemes

Let $\Omega = [a, b]$. We want to determine a function u such that

$$-u'' = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Consider the splitting $\Omega = \cup_{i=1}^N I_i$, $I_i = [x_{i-1}, x_i]$ of the interval $[a, b]$ and let $\mathscr{T}_h := \{I_i \mid i = 1, \ldots, N\}$, where $h = \max|I_i|$. For a given integer $k \geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathscr{T}_h$, i.e.

$$V_h := \big\{ v = \{v_0, v_b\} : v_0|_{I_i} \in P_k(I_i),\ v_b|_{x_i} \in \mathbb{R},\ v_b|_{x_0} = v_b|_{x_N} = 0 \big\}. \tag{2.3}$$

For $v \in V_h$, a weak derivative $D_w v$ is a piecewise polynomial such that on each I_i, $D_w v \in P_{k+1}(I_i)$ satisfies the relations

$$(D_w v, q)_{I_i} = -(v_0, Dq)_{I_i} + \langle v_b, q \rangle_{\partial I_i} \quad \text{for all} \quad q \in P_{k+1}(I_i), \tag{2.4}$$

where $Dv = dv/dx$ and $\langle v, w \rangle_{\partial I_i} = v(x_i)w(x_i) - v(x_{i-1})w(x_{i-1})$.

The SFWG method for the problem (2.1)-(2.2) consists in finding of $u_h \in V_h$ such that

$$(D_w u_h, D_w v) = (f, v_0) \quad \text{for all} \quad v = \{v_0, v_b\} \in V_h. \tag{2.5}$$

In what follows, we adopt the following notations:

$$(v, w)_{\mathscr{T}_h} = \sum_{i=1}^N (v, w)_{I_i} = \sum_{i=1}^N \int_{I_i} vw dx,$$