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Comparative Study of Space Iteration Methods Based on Nonconforming Finite Element for Stationary Navier-Stokes Equations

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Abstract. Steady Navier-Stokes equations are solved by three different space iteration methods based on the lowest order nonconforming finite element pairs $\mathscr{P}_1\mathscr{N}\mathscr{C}$-$\mathscr{P}_1$, including simple, Oseen, and Newton iterative methods. The stability and convergence of these methods are studied, and their CPU time and numerical convergence rate are discussed. Numerical results are in good agreement with theoretical findings. In particular, numerical experiments show that for large viscosity, the Newton method converges faster than to others, whereas the Oseen method is more suitable for the equations with small viscosity.

AMS subject classifications: 35L70, 65N30, 76D06

Key words: Navier-Stokes equations, nonconforming finite element, space iterative method, stability and convergence.

1. Introduction

Finite element methods are high-efficiency numerical methods. There are numerous works devoted to approximate solutions to initial and boundary value problems for partial differential equations [1, 7, 10, 14, 17, 19, 21, 22, 24, 27, 29, 35]. In particular, stationary Navier-Stokes equations have been considered in [3, 9, 13, 15, 18, 23, 25, 31–34]. Tone [39] studied error estimates for stationary Navier-Stokes equations based on a second order semi-implicit scheme. Girault and Raviart [6] provided theoretical analysis and solving algorithm for Navier-Stokes equations. However, up to now the main efforts have been spent on the study of mixed methods with finite elements of the lowest order [16, 26]. On the other hand, Crouzeix and Raviart [5] introduced linear nonconforming finite elements. They are devoted to a general finite element approximation of the solution of the Stokes equations for an incompressible viscous fluid. Nonconforming finite element methods have been studied and it was found that various simplistic elements are well suited for the numerical treatment of the incompressibility condition. Thus two-level/multi-level

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stabilized finite element methods for the stationary Navier-Stokes equations are considered in [11, 12, 28]. It was shown that these methods have optimal error estimates. Liu *et al.* [36] applied a weak Galerkin finite element method to Navier-Stokes equations and derived optimal-order error estimates in the H^1-norm for the velocity, and L^2-norm estimates for the velocity and the pressure.

Note that unlike the traditional finite element methods, the lowest order finite element pairs are simpler and more efficient in practice [2, 37]. In recent years, various locally stabilized mixed finite element methods have been developed for Navier-Stokes equations approximated by the lowest order finite element pairs [23].

The objective of this paper is to combine a locally stabilized mixed finite element method [26] with three iterative methods [8]. These methods are based on the $\mathscr{P}_1\mathscr{N}\mathscr{C}$-$\mathscr{P}_1$ approximations of the velocity and the pressure, and on the stabilization of these lowest-equal order finite elements through the residual of two local Gauss integrals at each element level. Moreover, special attention is paid to the treatment of the nonlinear terms occurring in the discrete Navier-Stokes equations that arise from these three iterative methods. Optimal estimates are obtained and numerical results completely support the theoretical estimates.

As we know, there are few results discussing the comparison of three iterative methods based on the nonconforming finite element method for the steady Navier-Stokes equations. Here, we prove that these three iterative methods converge with first order and second order. In numerical simulations, we compare these three iterative methods in terms of CPU time and convergence order for different viscosities. Among these three iterative methods, Methods I-III are valid in the case of larger viscosity for the driven cavity problem. Meanwhile, Method II can solve the problem with smaller viscosity as well.

This paper is structured as follows. In Section 2, some Sobolev spaces and continuous variational form are introduced. In Section 3, the local stabilized nonconforming finite element pairs are constructed. In Sections 4 and 5, the stability and convergence of three iterative methods are proved. Finally, numerical simulations verify theoretical results in Section 6.

2. Preliminaries

We assume that $\Omega \in R^d$, $d = 2, 3$ is an open bounded domain with the Lipschitz continuous boundary $\partial\Omega$ and consider the Navier-Stokes equation

$$\begin{aligned}
&\alpha u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f && \text{in } \Omega, \\
&\nabla \cdot u = 0 && \text{in } \Omega, \\
&u = 0 && \text{on } \partial\Omega,
\end{aligned} \tag{2.1}$$

where $\alpha \geq 0$ is a real number, $\nu > 0$ the viscosity, f an external force, and $u = (u_1, u_2, \ldots, u_d)$ and p denote the velocity and pressure fields, respectively.

Let $W^{s,r}(\Omega)$ denote the standard Sobolev space [4, 20, 30]. Besides, we write $H^s(\Omega)$ for $W^{s,2}(\Omega)$ and $\|\cdot\|_s$ and $|\cdot|_s$ for the standard norm and the semi-norm on $H^s(\Omega)$, respectively.