East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.011221.110122

Vol. **12**, No. 3, pp. 649-672
August 2022

A Multiple Interval Chebyshev-Gauss-Lobatto Collocation Method for Multi-Order Fractional Differential Equations

Shan Li, Guilei Sun, Yuling Guo and Zhongqing Wang*

University of Shanghai for Science and Technology, Shanghai 200093, China.

Received 1 December 2021; Accepted (in revised version) 11 January 2022.

Abstract. A multiple interval Chebyshev-Gauss-Lobatto collocation method for solving multi-order fractional differential equations is proposed. The hp-version error estimates of the Chebyshev spectral collocation method are obtained in L^2- and L^∞-norms. Numerical experiments illustrate the theoretical results.

AMS subject classifications: 34A08, 65N35, 41A10, 41A25

Key words: Multi-order fractional differential equation, Chebyshev-Gauss-Lobatto collocation method, hp-version error bound.

1. Introduction

In recent years, fractional differential equations (FDEs) have attracted great attention of physicists, mathematicians and engineers. The most important advantages of the FDEs are a nonlocal property and history dependence, which enable to provide a better description of the nature of memory and hereditary in physical phenomena and practical problems. It is worth noting that multi-order FDEs are more accurate than FDEs with a single fractional derivative term in practical simulations related to viscoelastic damping, fluid flows and quantum gravity [9, 16]. Nevertheless, unlike to single-order FDEs, it is more difficult to solve multi-order ones since they have a more complex history dependence.

There are various numerical methods developed for multi-order FDEs [2, 3, 5, 6, 12, 15, 17, 18, 20]. Although these numerical methods are often very effective, our goal here is to obtain more accurate numerical solutions. In particular, the spectral methods such as global high precision numerical methods, are quite suitable to solve the multi-order FDEs and a number of spectral methods have been investigated in the last decade. For example, spectral tau methods have been applied to multi-order time-space FDEs with Dirichlet boundary conditions in [1] and a multi-order time-fractional diffusion equations in [24]. For linear multi-order FDEs, a spectral tau method based on Müntz-Legendre polynomials was developed by Mokhtary *et al.* [14]. Besides, Lischke *et al.* [11] studied a Laguerre Petrov-Galerkin

*Corresponding author. *Email address:* `zqwang@usst.edu.cn` (Z. Wang)

spectral method for multi-order fractional initial value problems on the half line. Recently, Dabiri and Butcher [4] proposed a framework of spectral collocation methods for multi-order FDEs. Zaky and Ameen [25] converted multi-order FDEs into an equivalent Volterra integral equation of the second kind and used them to developed a spectral collocation method. The above mentioned methods are quite accurate in approximating of smooth solutions. However, the solutions of fractional problems are weakly singular at $t=0$, even if the source terms are very smooth [10]. Multiple interval spectral collocation methods can adopt flexibly the time step size and the local approximation orders to approximate the solution with the desired accuracy. Hence, they are effective in dealing with weakly singular problems.

Recently, Guo and Wang [7, 8] developed a Chebyshev collocation method for single fractional-order boundary value problems in and a Legendre spectral collocation method for the multi-order fractional differential equations and investigated the hp-convergence of the methods. Following their ideas, we consider a multiple interval Chebyshev-Gauss-Lobatto collocation method for the following multi-order fractional initial value problem:

$$\begin{aligned} &D^{\alpha+1}u+\sum_{i=1}^{r}a_iD^{\alpha_i}u=f\big(t,u(t)\big), \quad t\in(0,T], \\ &u(0)=0, \quad u'(0)=0, \end{aligned} \tag{1.1}$$

where $\alpha\in(0,1)$, $0<\alpha_1<\alpha_2<\cdots<\alpha_r<\alpha$, $f(t,u(t)):[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function which satisfies the Lipschitz condition

$$|f(s,y_1)-f(s,y_2)|\le L|y_1-y_2|. \tag{1.2}$$

Besides, D^ν is the Caputo derivative of order ν defined by

$$D^\nu u=\frac{1}{\Gamma(n-\nu)}\int_0^t(t-\tau)^{n-\nu-1}u^{(n)}(\tau)\mathrm{d}\tau, \quad \mathrm{t}>0, \quad \mathrm{n}-1<\nu<\mathrm{n}.$$

The method introduced here is based on the following Volterra integral equation — cf. [25]:

$$u(t)=\frac{1}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha}f\big(s,u(s)\big)ds-\sum_{l=1}^{r}\frac{a_l}{\Gamma(1+\alpha-\alpha_l)}\int_0^t(t-s)^{\alpha-\alpha_l}u(s)ds. \tag{1.3}$$

The main feature and contribution of this work are as follows.

- A multiple interval Chebyshev-Gauss-Lobatto collocation method is considered for multi-order FDEs. The desired accuracy of the numerical solution can be obtained by adjusting the time step size and the local approximation order.
- In order to improve the computational efficiency, one can exploit the fast Chebyshev transform with the Chebyshev-Gauss-Lobatto collocation points and the corresponding weights. This efficiently reduces the loss of potential precision.