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Analysis and Numerical Approximation for a Nonlinear Hidden-Memory Variable-Order Fractional Stochastic Differential Equation

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Abstract. We prove the wellposedness of a nonlinear hidden-memory variable-order fractional stochastic differential equation driven by a multiplicative white noise, in which the hidden-memory type variable order describes the memory of a fractional order. We then present a Euler-Maruyama scheme for the proposed model and prove its strong convergence rate. Numerical experiments are performed to substantiate the theoretical results.

AMS subject classifications: 60H20, 65L20

Key words: Variable-order fractional stochastic differential equation, hidden memory, Euler-Maruyama method, strong convergence.

1. Introduction

Stochastic differential equations (SDEs) provide a prominent modeling tool for many stochastic phenomena in sciences and engineering like biology, physics, chemistry and finance [6–8, 11, 15, 16, 19, 20, 24, 32]. In the processes containing nonlocal or memory effects, fractional derivatives provide a better description than integer-order derivatives do, which leads to the fractional SDEs (fSDEs). However, there is a large class of physical, biological and physiological diffusion phenomena that relate processes exhibiting accelerating or decelerating diffusion behaviors that cannot be characterized by the constant-order fractional diffusion equations. Typical features of these phenomena are that they are complex to analysis and the diffusion behavior depends on the time evolution, space variation

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or even system parameters. Since the orders of fractional derivatives in fSDEs are closely related to the fractal dimension of the media determined via the Hurst index [22], the variable fractional order derivatives are introduced to accommodate the structure change of the surroundings, which in turn leads to the variable-order (VO) fSDEs [12–14, 31, 33, 36, 37]. Works [9, 17, 27] introduced the space dependent VO into differential equations under the assumption that the probability density function is space dependent in the continuous time rand walk, which indicates that the memory rate depends on the space location in the considered system. Papers [27–29] proved that the mean square displacement is $\langle x^2(t)\rangle \propto t^{\alpha(t)}$, where $\alpha(t)$ is the order of the fractional diffusion equation. Measurement data also show that the diffusion behavior changing with the time evolution can be modeled by a time dependent VO fractional model. Thereby, it is more reasonable to investigate the VO fractional equations, and so further theoretical and numerical investigations of variable-order fSDEs are required for describing more complicated stochastic diffusion process.

Motivated by the preceding discussions, we study the following nonlinear Caputo fractional SDE with a hidden-memory variable order:

$$du = \big(-\lambda\, {}_0^C D_t^{\alpha(t)} u + f(t,u)\big)dt + b(t,u)dW, \quad t \in (0,T], \quad u(0) = u_0. \tag{1.1}$$

Here $\lambda \geq 0$, $0 \leq \alpha(t) \leq \alpha^* \leq 1/2$, and the hidden-memory variable-order fractional differential operator ${}_0^C D_t^{\alpha(t)}$ is defined in terms of the corresponding fractional integral via the Gamma function Γ [21, 28, 34, 35]

$${}_0^C D_t^{\alpha(t)} g(t) := {}_0 I_t^{1-\alpha(t)} g'(t), \quad {}_0 I_t^{1-\alpha(t)} g(t) := \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} g(s)ds. \tag{1.2}$$

Note that in the fractional integral, the power α assumes its historical state at the historical time instant s, which represents the memory of the order history and is named as hidden memory in order to distinguish it from the fading memory property of the fractional operators [27, 28].

FSDEs have attracted extensive attentions mathematically and numerically [1, 3–5, 10, 23, 25, 26, 36], while the corresponding investigations for variable-order FSDEs are meager. In a very recent work the well-posedness of a variable-order FSDE was analyzed, in which the variable-order fractional derivative is defined by (1.2) with $\alpha(s)$ replaced by $\alpha(t)$. Note that in such definition, the kernel becomes $(t-s)^{-\alpha(t)}/\Gamma(1-\alpha(t))$, which can be integrated into a close-form expression that significantly facilitates the mathematical analysis. However, the definition (1.2) does not enjoy this benefit, which shows the salient feature of the hidden-memory variable-order fractional problems and complicates the corresponding mathematical and numerical analysis.

We aim to prove the existence and uniqueness of the strong solution for (1.1), based on which we propose a Euler-Maruyama approximation and prove its optimal error estimates. The rest of this paper is organized as follows. In Section 2 we present preliminaries and the reformulation of the problem be used subsequently. In Section 3 we prove the wellposedness and moment estimate of the governing equation (1.1). In Section 4 we establish the