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A Semi-Tensor Product of Tensors and Applications

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Abstract. A semi-tensor product of matrices is proposed as a generalization of usual matrix product in the case where the dimensions of two factor matrices do not match. The properties of the semi-tensor product of tensors and swap tensors based on the Einstein product are studied. Applications of this new tensor product in image restoration and in finite dimensional algebras are discussed.

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1. Introduction

In recent years various concepts of matrix theory including eigenvalues, multi-linear systems, tensor decompositions, data mining, degree theory have been extended to problems involving tensors [1–3, 8, 9, 11, 16, 17, 23]. The semi-tensor product of matrices proposed by Cheng [5] found applications in the control design of dynamic systems, finite automata, graph theory, differential geometry, algebra, and data science [5–7, 15].

Definition 1.1 (cf. Cheng [5], Cheng & Zhang [7]). *Let* $\mathbf{x} = [x_1, \ldots, x_s]^T \in \mathbb{R}^s$, $\mathbf{y} = [y_1, \ldots, y_t]^T \in \mathbb{R}^t$.

(1) If $s = t \cdot n$, $n \in \mathbb{Z}_+$, we split $\mathbf{x}^T$ into t equal blocks, $\mathbf{x}_1^T, \ldots, \mathbf{x}_t^T$. Each block is an n-dimensional row vector. The (left) semi-tensor product of $\mathbf{x}^T$ and $\mathbf{y}$ is the n-dimensional row vector defined by

$$\mathbf{x}^T \ltimes \mathbf{y} := \sum_{k=1}^{t} y_k \mathbf{x}_k^T \in \mathbb{R}^{1\times n}.$$

(2) If $t = s \cdot n$, $n \in \mathbb{Z}_+$, we split $\mathbf{y}$ into s equal blocks $\mathbf{y}_1, \ldots, \mathbf{y}_s$. Each block is an n-dimensional column vector. The (left) semi-tensor product of $\mathbf{x}^T$ and $\mathbf{y}$ is the n-dimensional column vector defined by

$$\mathbf{x}^T \ltimes \mathbf{y} := \sum_{k=1}^{s} x_k \mathbf{y}_k \in \mathbb{R}^n.$$

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Definition 1.2 (cf. Cheng [5], Cheng & Zhang [7]). *Let $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{p\times q}$. If n is a divisor of p or p is a divisor of n, then the (left) semi-tensor product $C = [C^{ij}]$ of M and N, denoted by $C = M \ltimes N$, is a matrix that consists of $m \times q$ blocks, where each block is defined by*

$$C^{ij} = M(i,:) \ltimes N(:,j), \quad i = 1,\dots,m, \quad j = 1,\dots,q.$$

For example, if $A \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^p = \mathbb{R}^{p\times 1}$, and n is a divisor of p, say $p = t \cdot n$, then $A \ltimes \mathbf{x} \in \mathbb{R}^{tm}$ is a column vector. If p is a divisor of n, say $n = s \cdot p$, then $A \ltimes \mathbf{x} \in \mathbb{R}^{m\times s}$ is an $m \times s$ matrix.

We also note the following properties of the semi-tensor products.

(a) Setting $t = s$ in Definition 1.1, we obtain the usual Euclidean inner product, while setting $n = p$ in Definition 1.2, we obtain the usual matrix product.

(b) The right semi-tensor product of matrices was introduced in [7]. However, it is analogous to Definition 1.2 and is not considered here.

Let I_n denote the $n \times n$ identity matrix and $\otimes_K$ the Kronecker product.

Lemma 1.1 (Cheng *et al.* [5–7]). *Let A, B, C be matrices such that the corresponding semi-tensor products are well defined. Then we have*

(i) *If $A \in \mathbb{R}^{m\times np}$, $B \in \mathbb{R}^{p\times q}$, then $A \ltimes B = A(B \otimes_K I_n) \in \mathbb{R}^{m\times nq}$.*

(ii) *If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{np\times q}$, then $A \ltimes B = (A \otimes_K I_p)B \in \mathbb{R}^{mp\times q}$.*

(iii) $(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C)$.

(iv) $A \ltimes (\alpha B + \beta C) = \alpha A \ltimes B + \beta A \ltimes C$, *where α and β are constants.*

(v) $(\alpha B + \beta C) \ltimes A = \alpha B \ltimes A + \beta C \ltimes A$, *where α and β are constants.*

(vi) $(A \ltimes B)^T = B^T \ltimes A^T$.

(vii) $(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}$, *where A and B are invertible.*

(viii) $\operatorname{tr}(A \ltimes B) = \operatorname{tr}(B \ltimes A)$, *where $\operatorname{tr}(M)$ denotes the trace of a square matrix M.*

Now we consider an $mn \times mn$ matrix $W_{[m,n]}$, which plays an important role in semi-tensor products — cf. [5]. It is a permutation matrix, called the swap matrix, and constructed in the following way. Let

$$(11, 12, \dots, 1n, \dots, m1, m2, \dots mn)$$

denote the columns of $W_{[m,n]}$ and

$$(11, 21, \dots, m1, \dots, 1n, 2n, \dots, mn)$$