East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.261021.230122

Vol. **12**, No. 4, pp. 715-760
November 2022

Inverse Scattering Transform for the Defocusing Manakov System with Non-Parallel Boundary Conditions at Infinity

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Received 26 October 2021; Accepted (in revised version) 23 January 2022.

Abstract. The inverse scattering transform (IST) for the defocusing Manakov system is developed with non-zero boundary conditions at infinity comprising non-parallel boundary conditions — i.e., asymptotic polarization vectors. The formalism uses a uniformization variable to map two copies of the spectral plane into a single copy of the complex plane, thereby eliminating square root branching. The "adjoint" Lax pair is also used to overcome the problem of non-analyticity of some of the Jost eigenfunctions. The inverse problem is formulated in term of a suitable matrix Riemann-Hilbert problem (RHP). The most significant difference in the IST compared to the case of parallel boundary conditions is the asymptotic behavior of the scattering coefficients, which affects the normalization of the eigenfunctions and the sectionally meromorphic matrix in the RHP. When the asymptotic polarization vectors are not orthogonal, two different methods are presented to convert the RHP into a set of linear algebraic-integral equations. When the asymptotic polarization vectors are orthogonal, however, only one of these methods is applicable. Finally, it is shown that, both in the case of orthogonal and non-orthogonal polarization vectors, no reflectionless potentials can exist, which implies that the problem does not admit pure soliton solutions.

AMS subject classifications: 35Q51, 35B10, 35C08, 37K10

Key words: Manakov system, inverse scattering transform, soliton.

1. Introduction

This work is concerned with the defocusing Manakov system — i.e. the two-component defocusing nonlinear Schrödinger equation, written as

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2\nu\|\mathbf{q}\|^2\mathbf{q} = \mathbf{0} \tag{1.1}$$

with $\nu = 1$ and non-zero boundary conditions (NZBC) at infinity, namely

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$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm. \tag{1.2}$$

Here, $\mathbf{q} = \mathbf{q}(x,t) = (q_1, q_2)^T$ is a two-component vector, $\|\cdot\|$ is the standard Euclidean norm, $\|\mathbf{q}_\pm\| = q_o > 0$, and subscripts x and t denote partial differentiation. Throughout, asterisk denotes complex conjugation, and superscripts T and $\dagger$ denote, respectively, matrix transpose and Hermitian conjugate — i.e., conjugate transpose. The trivial space-independent phase rotation $\mathbf{q}'(x,t) = \mathbf{q}(x,t)\,\mathrm{e}^{2i\nu q_o^2 t}$ maps (1.1) into

$$i\mathbf{q}'_t + \mathbf{q}'_{xx} - 2\nu\big(\|\mathbf{q}'\|^2 - q_o^2\big)\mathbf{q}' = \mathbf{0}. \tag{1.3}$$

The asymptotic values $\mathbf{q}'_\pm = \mathbf{q}_\pm\,\mathrm{e}^{2i\nu q_o^2 t}$ are then independent of time, as long as $\mathbf{q}'_{xx}$ vanishes as $x \to \pm\infty$. Hereafter we will work with (1.3), but we will omit primes for brevity.

The scalar — i.e., one-component reduction of (1.3) is of course the celebrated nonlinear Schrödinger (NLS) equation. The NLS is a fundamental physical system, since it is a universal model governing the time evolution of quasi-monochromatic, weakly nonlinear dispersive wave trains [9,17]. As such, it arises in a bewildering variety of physical applications, ranging from deep water waves, to optics, acoustics, condensed matter (Bose-Einstein condensates) and beyond [5,6,30,39]. The Manakov system (1.3) is also a physically relevant model, governing the time evolution of coupled quasi-monochromatic waves in optics as well as Bose-Einstein condensates.

It is well known that the scalar NLS equation is a completely integrable system, as shown by Zakharov and Shabat [49] (see also [3,5,38]). Shortly afterwards, it was shown in [37] that the two-component generalization of the NLS equation, namely the system (1.3), is also integrable, and that the initial-value problem can be solved by way of the inverse scattering transform (IST). However, only the case of localized initial conditions, namely $q_o = 0$, was studied initially. The IST for the scalar defocusing nonlinear Schrödinger equation with NZBC at infinity — i.e., $q_o \neq 0$, was also formulated early on by Zakharov and Shabat [50] (see also [15,22,23,25]), but the generalization to the Manakov system remained an open problem for many decades.

Following some earlier results [27], a full formulation of the IST for the defocusing Manakov system with NZBC was finally presented in [40]. The problem was then revisited in [12] and generalized to defocusing coupled nonlinear Schrödinger systems with more than two components in [13,41]. The IST for the focusing two-component case with NZBC was also formulated in [35].

In all of the above works, however, including those on the defocusing Manakov system, only the special case in which $\mathbf{q}_\pm = \mathbf{q}_o\,\mathrm{e}^{i\theta_\pm}$, with $\theta_\pm \in \mathbb{R}$ was solved. We refer to this as the case of parallel NZBC. In this work we generalize the IST (and in particular the inverse problem) to the case of non-parallel NZBC, namely arbitrary vectors $\mathbf{q}_\pm$ subject to the only constraint $\|\mathbf{q}_\pm\| = q_o$. (To avoid confusion, we should emphasize that a large part of the formulation of the direct problem in [12,35,40] also carries over directly to the more general non-parallel case. However, the formulation of the inverse problem in those works does not, and indeed the requirement that $\mathbf{q}_+$ and $\mathbf{q}_-$ are parallel plays a key role in the normalization of the Riemann-Hilbert problem in those works.)