East Asian Journal on Applied Mathematics Vol. **12**, No. 4, pp. 821-847
doi: 10.4208/eajam.070921.150222 November 2022

Numerical Study of Time-Fractional Telegraph Equations of Transmission Line Modeling

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Received 7 September 2021; Accepted (in revised version) 15 February 2022.

Abstract. The stability and uniqueness of the solutions of time-fractional telegraph equations arising in the transmission line modeling are proved. The corresponding initial-boundary problems are then solved by a finite difference scheme. It is shown that the scheme is unconditionally stable and convergent. Computational efficiency of the method can be enhanced by transforming it into two finite volume schemes for solving two uncoupled time-fractional convection equations. Numerical experiments validate the theoretical results and show the efficiency of this approach even for the problems the solutions of which are not smooth at the initial moment.

AMS subject classifications: 65M10, 78A48

Key words: Finite difference scheme, time-fractional telegraph equation, transmission line modeling, non-smooth initial value.

1. Introduction

The telegraph equation is the pair of the following coupled linear partial differential equations that describe the voltage $V(x,t)$ and current $I(x,t)$ on an electrical transmission line with distance and time:

$$
\begin{aligned}
&V_x(x,t)+LI_t(x,t)+RI(x,t)=0,\\
&I_x(x,t)+CV_t(x,t)+GV(x,t)=0,
\end{aligned}
$$

where R,L,C, and G are respectively the distributed resistance, the distributed inductance, the capacitance, and the conductance. Traditional telegraph equations neglect the accumulation of electrical charge along the line, memory effects in polarization, and magnetization processes.

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In order to include the effects of charge accumulation along the line, Cvetićanin *et al.* [4] introduced a generalized time-fractional telegraph equation with the hypothesis that capacitive and inductive phenomena can display hereditary effects modeled by fractional calculus — viz.

$$\begin{aligned}&\bar{\tau}\,{}_0^CD_t^{\beta}U'(x,t)+U'(x,t)-RI(x,t)=0,\\&\bar{C}\,{}_0^CD_t^{\gamma}U(x,t)+GU(x,t)+I_x(x,t)=0,\\&\bar{L}\,{}_0^CD_t^{\alpha}I(x,t)+U'(x,t)+U_x(x,t)=0,\end{aligned}\tag{1.1}$$

where $\alpha,\beta,\gamma\in(0,1)$ are parameters and ${}_0^CD_t^{\alpha}w(t)$ is the Caputo fractional derivative defined by

$${}_0^CD_t^{\alpha}w(t):=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}w'(s)ds.$$

In the Eqs. (1.1), $\bar{\tau}$ is the degree of the influence of the charge accumulation along the line measured in $\Omega Fs^{\beta-1}$, $\bar{C}$ the fractional capacitance measured in $Fs^{\gamma-1}$, and $\bar{L}$ the fractional inductance measured in $Hs^{\alpha-1}$. The generalized time-fractional telegraph equation (1.1) can be used to describe the propagation of high-frequency signals in small-scale systems such as CMOS on-chip and microstrip lines.

In this work, we study the numerical solutions of the following simplified model with $\bar{\tau}=0$ and $\alpha=\beta$ on $\Omega=[-1,1]\times[0,T]$:

$$\begin{aligned}&{}_0^CD_t^{\alpha}u(x,t)+a_1u(x,t)+b_1v_x(x,t)=f(x,t),\\&{}_0^CD_t^{\alpha}v(x,t)+a_2v(x,t)+b_2u_x(x,t)=g(x,t),\end{aligned}\tag{1.2}$$

where $\alpha\in(0,1)$ is a parameter and the coefficients $a_1,a_2,b_1,b_2>0$ are constants.

We note that other types of time-fractional telegraph equations have been employed in [12,15,23] in order to simulate heat conduction in rigid conductors, in porcine muscle and blood, and to govern iterated Brownian motions and telegraph processes with Brownian time. However, analytical solutions of time-fractional telegraph equations are relatively complex and are rarely available. Therefore, numerical methods have to be used and there are many algorithms developed to solve them — cf. Refs. [3,5,9,13,18,19,22]. In particular, let us note the methods using formulas based on a local interpolation for the discretization of the Caputo fractional derivative such as the $\mathscr{L}1$ formula based on linear interpolation [16], the $\mathscr{L}1\_2$ formula based on linear and quadratic interpolations [6], and the $\mathscr{L}2\_1_\sigma$ formula [2]. Another formula — viz. $\mathscr{F}\mathscr{L}2\_1_\sigma$ in [20], uses the $\mathscr{L}2\_1_\sigma$ formula and a fast algorithm [8] and can speed up the evaluation of the Caputo fractional derivative. In [10,14], a discrete fractional Grönwall inequality [11] is applied to the study of the stability and convergence of approximation methods for linear reaction-subdiffusion equations with the Caputo derivative discretized by $\mathscr{L}1$ and $\mathscr{L}2\_1_\sigma$ formulas.

In this work, we use the $\mathscr{L}2\_1_\sigma$ discretization formula on uniform time mesh and obtain a stable and convergent finite difference scheme for the time-fractional telegraph (1.2) with smooth or non-smooth initial values. The stability and convergence of method involves the