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Boundedness in a Forager-Exploiter Model Accounting for Gradient-Dependent Flux-Limitation

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Abstract. The forager-exploiter model with gradient-dependent flux-limitation

$$
\begin{aligned}
u_t &= \Delta u - \chi \nabla \cdot \left(u k_f \left(1 + |\nabla w|^2\right)^{-\frac{\alpha}{2}} \right) \nabla w),\\
v_t &= \Delta v - \xi \nabla \cdot \left(v k_g \left(1 + |\nabla u|^2\right)^{-\frac{\beta}{2}} \right) \nabla u),\\
w_t &= \Delta w - (u+v)w - \mu w + r(x,t)
\end{aligned}
$$

is considered in smooth bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$. It is shown that if $\alpha > (N-2)/N(N-1)$, $\beta > 0$, then for any nonnegative functions $u_0, v_0, w_0 \in W^{2,\infty}(\Omega)$ such that $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, the problem has a global classical solution $(u,v,w) \in (C^0(\overline{\Omega} \times [0,\infty)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty)))^3$.

AMS subject classifications: 35A09, 35K65, 92C17

Key words: Chemotaxis, forager-exploiter model, boundedness, flux-limitation.

1. Introduction

This work is devoted to the forager-exploiter model proposed by Tania *et al.* [27] in order to study resource consumption. This model considers ecosystems consisting of two predator species — viz. the foragers who search for food independently and the exploiters who follow the foragers to get food/nutrient/prey indirectly. In addition to random diffusions, the taxis cascade model for food consumption in two populations assumes that the foragers directly orient their movements upward the gradients of the food concentration,

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whereas the exploiters use a parasitic strategy by tracking higher forager densities. The proliferation and death of foragers and exploiters are related.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\partial\nu$ denote the differentiation with respect to the outward normal vector ν on $\partial\Omega$. The original version of the forager-exploiter model with homogeneous Neumann boundary conditions and initial data has the form

$$\begin{aligned}
&u_t = \Delta u - \chi_1 \nabla\cdot(u\nabla w), && x\in\Omega, \quad t>0,\\
&v_t = \Delta v - \chi_2 \nabla\cdot(v\nabla u), && x\in\Omega, \quad t>0,\\
&w_t = \Delta w - \lambda(u+v)w - \mu w + r(x,t), && x\in\Omega, \quad t>0,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, && x\in\partial\Omega, \quad t>0,\\
&u(x,0)=u_0(x),\ \ v(x,0)=v_0(x),\ \ w(x,0)=w_0(x), && x\in\Omega,
\end{aligned} \tag{1.1}$$

where $\chi_1 > 0$ and $\chi_2 > 0$ are the sensitivity coefficients, u and v the sought densities of the forager and exploiter populations, respectively, $w(x,t)$ is the density of their nutrients, $-\lambda(u+v)w$ the consumption of chemoattractant, μ the degradation rate of nutrient, and $r(s,t) \geq 0$ a given production rate of the nutrient.

There are numerous studies of global existence, boundedness and the asymptotic behaviour of the solutions of the model (1.1). Under a smallness condition linking the initial data and the production rate, the global existence and the long time behavior of generalized solutions are established in [40]. In the one dimensional case, the solvability and global stability of the classical solutions of (1.1) have been investigated in [28]. In higher dimensions — i.e. if $N \geq 2$, J. Wang and M. Wang [32] showed that smallness of the initial data and the production rate (or smallness of taxis coefficients) ensures the global existence and the boundedness of the classical solutions. If generalized logistic sources are involved, Black [7] and J. Wang and M. Wang [32] suggested degradation rate conditions, which guaranty the existence of global generalized and classical solutions in two dimensions. Moreover, Wang [30] discovered that nonlinear diffusion can exclude blow up phenomenon. Recently, Cao [9] has found a global radial renormalized solution of the forager-exploiter model with singular sensitivities. For more results, we can refer the reader to Refs. [6,20,23,29,44,46].

Keller and Seger [18] introduced a chemotaxis model considering the aggregation of cellular slime mold toward a higher concentration of a chemical signal — viz.

$$\begin{aligned}
&u_t = \Delta u - \nabla\cdot(\chi u\nabla v), && x\in\Omega, \quad t>0,\\
&v_t = \Delta v - v + u, && x\in\Omega, \quad t>0.
\end{aligned} \tag{1.2}$$

The mathematical analysis of (1.2) and its variants mainly focuses on boundedness, blow-up, and the asymptotic behaviour of the solutions [3, 8, 10, 16, 19, 21, 36, 38]. As far as the associated Neumann initial-boundary value problem is concerned, it was shown that under a smallness condition on the initial data, the system (1.2) has a global bounded classical solution. On the other hand, many processes — e.g. nonlinear porous medium