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A Partially Greedy Randomized Extended Gauss-Seidel Method for Solving Large Linear Systems

Ai-Li Yang[1,2] and Xue-Qi Chen[3,*]

[1]*School of Mathematics and Statistics, Hainan Normal University, Haikou 570000, China.*
[2]*MOE Key Laboratory of Data Science and Intelligence Education, Hainan Normal University, Haikou 570000, China.*
[3]*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China.*

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Abstract. A greedy Gauss-Seidel based on the greedy Kaczmarz algorithm and aimed to find approximations of the solution $A^{\dagger}b$ of systems of linear algebraic equations with a full column-rank coefficient matrix A is proposed. Developing this approach, we introduce a partially greedy randomized extended Gauss-Seidel method for finding approximate least-norm least-squares solutions of column-rank deficient linear systems. The convergence of the methods is studied. Numerical experiments show that the proposed methods are robust and efficient.

AMS subject classifications: 65F10

Key words: Systems of linear equations, least-squares solution, randomized extended Gauss-Seidel method, convergence.

1. Introduction

Consider the system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$ is a large sparse rectangular matrix and $b \in \mathbb{R}^m$ a given vector, and note that for such systems, iteration methods are used more often than direct ones — cf. Refs. [2, 4, 12, 20]. In particular, for consistent linear systems (1.1), the Kaczmarz algorithm [27] is a very popular iterative projection method. It is a special case of row-action methods [1,12,14,15], and in the field of image processing it is known as the algebraic reconstruction technique [15, 23].

*Corresponding author. *Email addresses:* XueqiChen0813@163.com (X.-Q. Chen), yangaili@hainnu.edu.cn, yangaili@lzu.edu.cn (A.-L. Yang)

In the Kaczmarz method, each iteration is formed by projecting the current point to a hyperplane formed by an active row. More specifically, if $A^{(i)}$ refers to the i-th row of the matrix A and $b^{(i)}$ is the i-th entry of the vector b, then the Kaczmarz method starting from an initial guess x_0, can be formulated as follows:

$$x_{k+1} = x_k + \alpha_k \frac{b^{(i_k)} - A^{(i_k)} x_k}{\|A^{(i_k)}\|_2^2} \left(A^{(i_k)}\right)^T, \quad k = 0, 1, 2, \ldots, \tag{1.2}$$

where $\{\alpha_k\} \subset (0, 2)$ are the so-called relaxation parameters and the control sequence $\{i_k\}$ is chosen cyclically as

$$i_k = (k \bmod m) + 1.$$

Thus the Kaczmarz algorithm is very simple and very suitable for the solution of large linear systems. Therefore, it is widely used in various fields, including computerized tomography [16,24,28,32], image reconstruction [19,21,22,25,26,33], signal processing [13,30] and so on. However, sometimes the Kaczmarz algorithm converges very slowly — cf. [3,6] and the references therein. To improve the convergence, Strohmer and Vershynin [34] proposed a randomized Kaczmarz (RK) algorithm with expected exponential rate of convergence. Unlike the Kaczmarz method, the RK algorithm chooses the control sequence $\{i_k\}$ from the set $[m] := \{1, 2, \ldots, m\}$ at random, with probability proportional to $\|A^{(i_k)}\|_2^2$. Theoretical analysis shows that the RK algorithm with $\alpha_k = 1$ has exponential convergence rate in expectation. In recent years, the interest on Kaczmarz type algorithms was reignited because their simplicity and acceptable efficiency. Thus Bai and Wu [7] introduced a more effective probability criterion and constructed a greedy randomized Kaczmarz (GRK) method. This probability criterion can grasp larger entries of the residual vector at each iteration, which makes the corresponding GRK method to converge significantly faster than the RK method, both theoretically and practically. Following the ideas of [7,8], Zhang [35] proposed a greedy Kaczmarz (GK) algorithm for consistent linear systems (1.1). In the GK algorithm, the iteration sequence $\{x_k\}$ is generated by the iteration scheme (1.2), whereas the control sequence $\{i_k\}$ is determined by the greedy rule

$$i_k = \arg\max_{1 \le i \le m} \frac{\left|b^{(i)} - A^{(i)} x_k\right|^2}{\|A^{(i)}\|_2^2}.$$

The relaxation parameter α_k is chosen as $\alpha_k = \tilde{\alpha}_k$ if the term

$$\tilde{\alpha}_k = \arg\min_{\alpha_k} \left\{ \sum_{\substack{j_k \in \psi_k \\ j_k \neq i_k}} \left|b^{(j_k)} - A^{(j_k)} x_{k+1}\right|^2 + \omega \left|b^{(i_k)} - A^{(i_k)} x_{k+1}\right|^2 \right\} \tag{1.3}$$

is located in a given subinterval of $(0, 2)$. Otherwise, $\alpha_k = 1$. In (1.3), ω is a given weight coefficient, ψ_k an index set consisting of the row indices corresponding to the larger entries of the relative residual vector of the linear system $Ax = b$.

If the coefficient matrix A of (1.1) is of full column-rank, Leventhal *et al.* [29] proposed a randomized Gauss-Seidel (RGS) algorithm (also called the randomized coordinate descent (RCD) algorithm) by applying the RK method to the normal equation $A^T Ax = A^T b$.