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# Error Estimates of Finite Difference Methods for the Fractional Poisson Equation with Extended Nonhomogeneous Boundary Conditions

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**Abstract.** Two efficient finite difference methods for the fractional Poisson equation involving the integral fractional Laplacian with extended nonhomogeneous boundary conditions are developed and analyzed. The first one uses appropriate numerical quadratures to handle extended nonhomogeneous boundary conditions and weighted trapezoidal rule with a splitting parameter to approximate the hypersingular integral in the fractional Laplacian. It is proven that the method converges with the second-order accuracy provided that the exact solution is sufficiently smooth and a splitting parameter is suitably chosen. Secondly, if numerical quadratures fail, we employ a truncated based method. Under specific conditions, the convergence rate of this method is optimal, as error estimates show. Numerical experiments are provided to gauge the performance of the methods proposed.

**AMS subject classifications**: 35R11, 65M06

**Key words**: Fractional Poisson equation, finite difference method, nonhomogeneous boundary condition, error estimates, integral fractional Laplacian.

## 1. Introduction

Fractional partial differential equations (FPDEs) provide an adequate and accurate description of various complex physical phenomena such as anomalous diffusion, memory behavior, long-range interaction and so on, which cannot be modeled properly by classical PDEs [3, 16]. Nowadays, FPDEs have been widely applied in various fields, including quantum mechanics [12], ground-water solute transport [4], stochastic dynamics [15] and finance [9].

A generalization of the classical Laplacian — viz. the fractional Laplacian operator can be defined in different ways [14]. In particular, the hypersingular integral fractional Laplacian operator attracted substantial attention. It has been extensively studied by many

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researchers over the last decades. Duo *et al.* [5] developed a finite difference method and proved that the convergence rate of the method depends on the solution regularity and a splitting parameter. Victor and Ying [21] used singularity subtraction for constructing a simple translation-invariant discretization scheme, which can be efficiently handled by fast Fourier transform. Hao *et al.* [10] studied a centered finite difference scheme for a fractional diffusion equation with the integral fractional Laplacian. Acosta *et al.* [1] investigated the regularity of the fractional Laplace equation and proved the optimal convergence of a linear finite element method on quasi-uniform and graded meshes.

We note that in practice, the integral fractional Laplacian with nonhomogeneous boundary condition is more useful. However, the presence of such boundary conditions leads to new problems such as the development of efficient numerical solvers and treatment of far field boundary conditions. Tang *et al.* [20] employed a rational basis in a spectral method for FPDEs with fractional Laplacian on unbounded domains and established optimal error estimates of the corresponding scheme. Xu *et al.* [24] used spherical means to develop an efficient algorithm for multi-dimensional integral fractional Laplacian. Wu *et al.* [23] proposed an efficient operator factorization method, where far field boundary conditions are approximated by numerical quadratures. Sun *et al.* [19] considered a finite difference method for nonhomogeneous fractional Dirichlet problem with compactly supported boundary conditions. Huang and Oberman [11] developed a finite difference-quadrature method with asymptotic approximations of extended Dirichlet boundary condition.

In this paper, we apply a finite difference method to the one-dimensional fractional Poisson equation with the integral fractional Laplacian

$$\begin{aligned} &(-\Delta)^s u(x) = g(x), && x \in (-L, L), \\ &u(x) = f(x), && x \in \mathbb{R} \setminus (-L, L), \end{aligned} \tag{1.1}$$

where $s \in (0,1)$ and $f(x)$ is a positive function decaying to zero as $x \to \pm\infty$. The fractional Laplacian $(-\Delta)^s$ in (1.1) is defined by

$$(-\Delta)^s u(x) := c_{1,s} \text{ P.V.} \int_{\mathbb{R}} \frac{u(x) - u(x')}{|x - x'|^{1+2s}} dx', \tag{1.2}$$

where P.V. denotes the principal value integral, and $c_{1,s}$ denotes the normalization constant

$$c_{1,s} = \frac{2^{2s} s \Gamma(1/2 + s)}{\pi^{1/2} \Gamma(1 - s)}.$$

Let us also recall that if $u(x)$ belongs to the Schwartz space of rapidly decaying functions, then the fractional Laplacian can also be defined by

$$(-\Delta)^s u(\xi) = \mathscr{F}^{-1}\big(|\xi|^{2s} \mathscr{F}(u)\big) \quad \text{for} \quad s > 0,$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are respectively the Fourier transform and its inverse [1].

Here we focus on the error estimates of the numerical methods under consideration. The far field boundary conditions are always assumed to be decreasing, which differs from many existing results in the literature. The main contribution of this work is twofold: