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Stability and Convergence of $L1$-Galerkin Spectral Methods for the Nonlinear Time Fractional Cable Equation

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Abstract. A numerical scheme for the nonlinear fractional-order Cable equation with Riemann-Liouville fractional derivatives is constructed. Using finite difference discretizations in the time direction, we obtain a semi-discrete scheme. Applying spectral Galerkin discretizations in space direction to the equations of the semi-discrete systems, we construct a fully discrete method. The stability and errors of the methods are studied. Two numerical examples verify the theoretical results.

AMS subject classifications: 35R11, 65M12, 65M70

Key words: Nonlinear fractional cable equation, spectral method, stability, error estimate.

1. Introduction

Fractional models provide a more detailed and comprehensive description of the memory, heredity, and non-locality. Therefore, fractional calculus is widely used in viscoelasticity and non-Newtonian fluid mechanics [28], fractional heat conduction [23], fractional Brownian models with stochastic volatility [20], and physics [6]. Theory and application of fractional calculus has gradually become a hot new issue [22].

In particular, recently the time fractional diffusion equations (TFDE) and fractional wave equations have been intensively studied both theoretically and numerically. Thus Schneider and Wyss [24, 31] analyzed fractional diffusion wave equations, whereas Sun *et al.* [27] investigated a fully discrete difference scheme for their solution. Liu *et al.* [15] studied the stability and convergence of discrete non-Markovian random walk approximations of TFDE obtained by a finite difference method. Langlands and Henry [7] established an

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implicit numerical scheme based on finite difference approximations for time fractional diffusion equations with the Riemann-Liouville fractional derivative. Sun *et al.* [26] used the Alikhanov's work [1], in order to study the stability and convergence of the discrete scheme for fractional wave equations. A new second-order midpoint approximation formula for the Riemann-Liouville derivative has been suggested in [3], and two implicit numerical methods for the fractional cable equation are considered in [16]. For time-fractional parabolic equations with nonsmooth solutions, Liu *et al.* [10] developed a numerical method based on thee change of variable $s = t^{\beta}$ and established an optimal error estimate of the $L1$ finite difference method. The unconditional stability and convergence of the fast difference scheme for a second-order multi-term time-fractional sub-diffusion problem are proved in [5].

Non-linear fractional equations and numerical methods of their solution are also studied. Thus high order methods for nonlinear fractional ordinary differential equations are developed in [12]. Liu *et al.* [17, 18] considered a finite element method combined with a finite difference scheme for a fourth-order nonlinear time fractional reaction-diffusion problem. Li *et al.* [9] considered $L1$-Galerkin FEMs for time-fractional nonlinear parabolic problems, whereas Duo and Zhang [4] studied numerical methods for the fractional nonlinear Schrödinger equation. The stability and convergence of an implicit numerical method for nonlinear fractional diffusion equations are analyzed in [13, 37], and finite element approximations for the nonlinear fractional Cable equation are discussed in [19, 29]. Li and Yi [8] constructed a discrete scheme for a two-dimensional nonlinear time-fractional subdiffusion equation, and Zhang and Jiang [36] developed an unconditional convergent numerical scheme for a two-dimensional nonlinear time fractional diffusion-wave equation. A compact difference scheme for nonlinear fourth order fractional sub-diffusion wave equation has been proposed in [21], and a linearised three-point combined compact difference method with weighted approximation for nonlinear time fractional Klein-Gordon equations is developed [35].

Spectral methods are important numerical tools in fractional differential equations [25]. Thus the spectral-collocation method for fractional integro-differential equations is studied in [33, 34]. Chen and Yang [32] considered a numerical method for nonlinear Volterra integro-differential equations. Wei and Chen [30] studied a Jacobi spectral approach to second kind multidimensional linear Volterra integral equations. Xu and Li [11] considered finite difference-spectral discretizations for the time fractional diffusion equation. A finite difference-spectral method for the fractional Cable equation is investigated in [14].

The fractional Cable equation is used in modelling of anomalous electro-diffusion in nerve cells. In the present work, we develop a numerical scheme for the nonlinear time fractional cable equation which is based on finite difference approximations in the time direction and a Galerkin spectral method in the space direction. The stability and the errors of the corresponding semi-discrete scheme are proved. Besides, we consider a fully discrete scheme and determine the related errors. Numerical examples verify the theoretical results.

This paper is organized as follows. In Section 2, we use finite difference approximations in the time direction and establish a semi-discrete scheme. After that, we employ Galerkin spectral approximations for the space direction and obtain a fully discrete scheme. Section 3 is devoted to the stability of the semi-discrete problem. The error analysis is presented in