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Convergence Rates of Split-Step Theta Methods for SDEs with Non-Globally Lipschitz Diffusion Coefficients

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Abstract. The present work analyzes the mean-square approximation error of split-step theta methods in a non-globally Lipschitz regime. We show that under a coupled monotonicity condition and polynomial growth conditions, the considered methods with the parameters $\theta \in [1/2, 1]$ have convergence rate of order 1/2. This covers a class of stochastic differential equations with super-linearly growing diffusion coefficients such as the popular 3/2-model in finance. Numerical examples support the theoretical results.

AMS subject classifications: 60H35, 60H15, 65C30

Key words: Stochastic differential equation, non-globally Lipschitz coefficient, split-step theta method, strong convergence rate.

1. Introduction

Stochastic differential equations (SDEs) play an important role in various fields of natural and social sciences. However, most of SDEs can not be solved analytically, so that numerical simulations become a vital tool for understanding SDE models. Various numerical schemes are developed, with strong and weak approximation errors well studied under the classical conditions that the coefficients of SDEs are globally Lipschitz continuous [21, 30]. However, since the majority of nonlinear SDEs arising in applications have super-linearly growing coefficients, the study of their numerical approximations is a non-trivial task. As is shown in [16], for a large class of SDEs with super-linearly growing coefficients the popularly used Euler-Maruyama (EM) method can produce numerical solutions with divergent moment bounds as the time step-size tends to zero. This results in strong and weak divergence of the numerical approximations. Such observations can be also found in the early reference [11, Section 3], where a motivating example was given. Note that a large number of works devoted to the numerical analysis of SDEs under non-globally Lipschitz conditions makes an emphasis on implicit schemes — cf. [1–3,9,10,12,13,19,23,26,27,34,36], and on

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developing approximation methods based on modifications of traditional explicit schemes — cf. [4–7, 14, 15, 17, 18, 20, 22, 24, 25, 29, 31–33, 35, 39], to just mention a few.

This work is concerned with a kind of split-step implicit schemes for SDEs with non-globally Lipschitz coefficients, where the drift and diffusion coefficients are assumed to obey the coupled monotonicity condition (2.3). This setting allows super-linearly growing diffusion coefficients and covers the popular 3/2-model in finance. The schemes under consideration are called the split-step theta (SST) methods — cf. the Eqs. (2.2) below. They have been introduced by Huang [13], where the exponential mean square stability of SST methods and the usual stochastic theta methods (STMs), was examined under the monotonicity condition (2.3). In particular, it was shown that the SST methods with $\theta > 1/2$ have better nonlinear stability properties than the STMs do. The SST methods extend the split-step backward Euler (SSBE) method proposed by Mattingly *et al.* [28], where the ergodicity of SDEs with locally Lipschitz coefficients and their approximations have been studied. They showed that the explicit EM method does not inherit the geometric ergodicity of such SDEs while the SSBE scheme was able to reproduce the ergodicity. The strong convergence rate of the SSBE scheme was first established in [9] for SDEs with non-globally Lipschitz drift but globally Lipschitz diffusion coefficients. Similar strong convergence results are derived in [38] for the SST methods with $\theta \in [1/2, 1]$ and in [19] for semi-implicit split-step numerical methods and globally Lipschitz continuous diffusion coefficients. If the diffusion coefficients can grow super-linearly, some efforts have been made to prove the strong convergence rate of split-step type methods. Thus Liu *et al.* [23] proposed a family of split-step balanced θ-methods for SDEs with non-globally Lipschitz continuous coefficients. Using the fundamental strong convergence theorem [33], they obtained the desired strong convergence rate. Besides, using the notions of stochastic C-stability and stochastic B-consistency, Andersson and Kruse [2] obtained the mean-square convergence rate of the SSBE scheme under the coupled monotonicity condition (2.3) for non-globally Lipschitz diffusion coefficients. However, to the best of the authors knowledge, in the case of non-globally Lipschitz diffusion coefficients, the convergence rates of general SST methods (2.2) with $\theta \in [1/2, 1]$ has not been studied. As pointed out in [9, p. 1060], the split-step implicit method with $\theta = 1/2$, may be of practical interest for Hamiltonian problems perturbed by damping and/or noise.

Motivated by the above results, we study the mean-square error of the general SST methods for SDEs with possibly super-linearly growing diffusion coefficients. In particular, we show that SST methods with $\theta \in [1/2, 1]$ converge with the rate 1/2 under a coupled monotonicity condition and polynomial growth conditions. This setting covers a class of SDEs with super-linearly growing diffusion coefficients including the popular 3/2-model in finance. Wang *et al.* [36] proposed a new approach to the mean-square error analysis for STMs. It does not require a priori high-order moment estimates of numerical approximations and allows to recover mean-square convergence rates of STMs with $\theta \in [1/2, 1]$ under the coupled monotonicity condition (2.3).

The present article extends the ideas of [36] to general SST methods with $\theta \in [1/2, 1]$. Unlike [36], we have to introduce an auxiliary process $\widetilde{X}(t_n)$ and develop a new techniques in the error analysis — cf. the proof of Theorem 3.1 and comments at the end of Section 3.