East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.291121.090722

Vol. **13**, No. 1, pp. 76-94
February 2023

A Class of Relaxed TTSCSP Iteration Methods for Weakly Nonlinear Systems

Xiaohui Yu and Qingbiao Wu*

School of Mathematical Sciences, Zhejiang University, HangZhou, 310027, Zhejiang, P.R.China.

Received 29 November 2021; Accepted (in revised version) 9 July 2022.

Abstract. A relaxed TTSCSP (RTTSCSP) iteration method for complex linear systems is constructed. Based on the strong dominance and separability of linear and nonlinear terms, Picard-RTTSCSP and nonlinear RTTSCSP-like iterative methods are developed and applied to complex systems of weakly nonlinear equations. The convergence of the method is investigated. Besides, optimal iterative parameters minimizing the upper bound of the spectral radius are derived. Numerical examples show the effectiveness and applicability of the methods to complex systems of weakly nonlinear equations.

AMS subject classifications: 65M10, 78A48

Key words: Picard-RTTSCSP method, nonlinear RTTSCSP-like method, weakly nonlinear system, convergence.

1. Introduction

Let $\mathbb{D}$ be an open convex subset of the n-dimensional complex linear space $\mathbb{C}^n$ and $\phi : \mathbb{D} \to \mathbb{C}^n$ a continuously differentiable nonlinear function. We consider the complex weakly nonlinear system

$$Ax := (W + iT)x = \phi(x), \tag{1.1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are real symmetric positive definite matrices and $i = \sqrt{-1}$ the imaginary unit. This system can be written as

$$F(x) := Ax - \phi(x) = 0,$$

and in what follows, we mainly use this form of the Eq. (1.1).

If the linear part A is strongly dominant over in a specific norm, then the system (1.1) is called weakly nonlinear — cf. [6, 10, 33]. Finding the solutions of systems (1.1) has numerous applications in engineering, nonlinear partial differential equations, saddle point problems in image processing and nonlinear optimization problems [5–7, 9, 16, 21]. As

*Corresponding author. *Email address:* qbwu@zju.edu.cn (Q. Wu)

far as the general nonlinear equation $F(x) = 0$, the most popular solution method is the second-order classical iteration Newton scheme — i.e.

$$x_{n+1} = x_n - F^{'}(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \ldots,$$

where $F : \mathbb{D} \to \mathbb{C}^n$ is a continuous differentiable function. Nevertheless, in practical computations it is difficult to construct the Jacobian matrix and derive an exact solution of the above equations. Therefore, the inexact Newton method

$$\begin{aligned} & x_{n+1} = x_n + s_n, \\ & F'(x_n)s_n = -F(x_n) + r_n, \\ & \|r_n\|/\|F(x_n)\| \leq \eta_n, \end{aligned}$$

where $\eta_n \in [0, 1)$, attracted a substantial attention — cf. [1, 3, 15, 20]. In particular, combining the Newton method with inner solvers allowed to adjust various numerical methods for linear systems to non-linear equations [1, 4, 8, 12–14, 23, 26, 28, 29, 31].

By adopting the HSS scheme as the internal solver, Bai [8] established a Newton-HSS methods for solving nonlinear systems with large sparse positive definite Jacobian matrices. Yang and Bai [10] proposed nonlinear HSS-like and the Picard-HSS iteration methods for weakly nonlinear systems with specific properties. Considering weakly nonlinear equations with large sparse matrices, Pu and Zhu [22] improved algorithms for linear equations and used them to develop generalized nonlinear compound splitting iterative methods, named nonlinear GPHSS-like and Picard-GPHSS. The solution of Toeplitz systems of weakly nonlinear equations have been studied by Zhu and Zhang [34], who developed nonlinear CSCS-like and Picard-CSCS iterative schemes, which are nonlinear composite iteration algorithms. Adopting the AIPCG iteration technique, Jiang and Guo [18] established Picard-AIPCG algorithms for solving the equations of this type. A class of lopsided PMHSS iteration methods and a nonlinear LHSS-like method converging to the unique solution, have been proposed by Li and Wu [19]. Zeng and Zhang [30] constructed a PTGHSS iteration scheme and two PTGHSS-based iteration methods for weakly nonlinear systems. Chen *et al.* [11] applied nonlinear DPMHSS-like and Picard-DPMHSS methods, based on double-parameter PMHSS iterative technique, to weakly nonlinear systems. Using an HSS scheme, Amiri [2] established a Jacobi-free HSS algorithm for weakly nonlinear systems. Combining accelerated GSOR and preconditioned GSOR method, Wu and Qi [27] introduced Picard-preconditioned GSOR and Picard-accelerated GSOR methods for weakly nonlinear equations with complex matrices.

Taking into account excellent properties and efficient performance of HSS-like iterative methods, Zheng *et al.* [32] applied a DSS iteration scheme to complex symmetric linear equations. Xie and Wu [28] adjusted the Newton-DSS method to nonlinear systems with large sparse complex symmetric Jacobian matrices. Besides, a complex Sylvester matrix equation has been solved by Feng and Wu [17] by the Lopsided DSS iteration method. In order to accelerate the DSS scheme, Siahkolaei *et al.* [24] established a two-parameter two-step scale-splitting (TTSCSP) method, which was then used in the solution of systems of weakly nonlinear equations by Siahkolaei *et al.* [25].