East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-064.210722

Vol. **13**, No. 2, pp. 230-245
May 2023

A Chebyshev Polynomial Neural Network Solver for Boundary Value Problems of Elliptic Equations

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Received 7 March 2022; Accepted (in revised version) 21 July 2022.

Abstract. A Chebyshev polynomial neural network for solving boundary value problems for one- and two-dimensional partial differential equations is constructed. In particular, the input parameters are expanded by Chebyshev polynomials and fed into the network. A loss function is constructed, and approximate solutions are determined by minimizing the loss function. Elliptic equations are used to test a Chebyshev polynomial neural network solver. The numerical examples illustrate the high accuracy of the method.

AMS subject classifications: 35Q68

Key words: Chebyshev polynomial, neural network, elliptic equation, deep learning.

1. Introduction

Various problems in physics and engineering can be formulated in terms of partial differential equations (PDEs). In particular, the equilibrium of physical systems are often described by elliptic equations. However, analytical solutions of the corresponding equations are rarely available, hence numerical methods have to be used. Various numerical techniques developed for PDEs include finite-difference, finite element, finite volume, and spectral methods [1, 28]. Besides, recent substantial growth of computer resources [25] led to rapid development of deep learning methods and the applications of artificial intelligence in language and image recognition [13, 16], signal processing, computer vision, and other areas.

Neural network has been also used in the numerical approximation. Thus Cybenko [3] proved that functions can be approximated by a single-layer feed-forward neural network with arbitrary accuracy — i.e. the theory of universal approximation. Hornik [8] showed that if the activation function is smooth and there are enough hidden cells, the feed-forward

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neural network can approximate any function. The potential of neural networks in solving the differential equations has been first demonstrated in Refs. [14, 15, 26]. Nowadays, there are numerous methods using the artificial neural networks in order to solve differential equations. In particular, Mall *et al.* [20] proposed a Chebyshev neural network based method for the second-order non-linear ordinary differential equations of the Lane-Emden type. Raissi *et al.* [24] designed a neural network of physical information for nonlinear PDSs. E *et al.* [6] constructed a deep Ritz method which can use a neural network for solving the variational formulation of differential equations. Sirignano *et al.* [29] designed a deep Galerkin method for high-dimensional partial differential equations. Karumuri *et al.* [11] developed deep neural networks for solving high-dimensional random elliptic PDEs. Verma *et al.* [30] discussed multi-layer perceptron artificial neural network technique for the solution of Lane-Emden type differential equations. In the last few years, there are growing literatures on neural network based numerical methods for PDEs [4, 27]. In particular, E [5] emphasised the important relationship between machine learning and computational mathematics. Jagtap *et al.* [10] proposed a generalized space-time domain decomposition framework for the physics-informed neural networks used for solving nonlinear PDEs on the domains with arbitrary complex-geometry. Liao *et al.* [18] proposed a deep Ritz method to deal with essential boundary conditions encountered in the deep learning-based numerical solvers for PDEs. Liu *et al.* [19] presented VPVnet – a deep neural network method for the Stokes equations with reduced regularity. Huang *et al.* [9] studied a novel deep learning method for variational problems with essential boundary conditions. Feng *et al.* [7] proposed an adaptive learning approach based on temporal normalizing flows for solving time-dependent Fokker-Planck equations.

In this work, we employ a Chebyshev polynomial neural network to solve boundary value problems for elliptic PDEs. Previously, the method has been used for approximating general functions [17] and ordinary differential equation [2]. It is developed in spirit of the Chebyshev Galerkin spectral method and can achieve the high accuracy of spatial approximations [1]. The specific steps are as follows. Firstly, the input parameters are defined by Chebyshev polynomials and fed into the network. Secondly, a loss function related to the corresponding PDE is constructed and grid points are chosen. After that, an approximate solution is found by minimizing the loss function constructed.

The paper is organized as follows. In Section 2, we introduce neural networks, describe their training and the approach to the solution of differential equations by feed-forward neural networks. In Section 3, a Chebyshev polynomial neural network is introduced and used in solving partial differential equations. Section 4 contains numerical examples for one- and two-dimensional elliptic equations. The examples show that the method can solve partial differential equations with a high accuracy. Our conclusions are in Section 5.

2. Introduction to Feed-Forward Neural Networks

A neural network is an abstract mathematical model simulating the operation of the human brain. It consists of many simple, interconnected processing units called neurons. The neural networks studied in this paper are composed of the same type of neurons in