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$L2$-1_σ Finite Element Method for Time-Fractional Diffusion Problems with Discontinuous Coefficients

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Abstract. A time-fractional diffusion equation with an interface problem caused by discontinuous coefficients is considered. To solve it, in the temporal direction Alikhanov's $L2$-1_σ method with graded mesh is presented to deal with the weak singularity at $t = 0$, while in the spatial direction a finite element method with uniform mesh is employed to handle the discontinuous coefficients. Then, with the help of discrete fractional Grönwall inequality and the robustness theory of $\alpha \to 1^-$, we show that the method has stable error bounds at $\alpha \to 1^-$, the fully discrete schemes $L^2(\Omega)$ norm and $H^1(\Omega)$ semi-norm are unconditionally stable, and the optimal convergence order is $\mathscr{O}(h^2 + N^{-\min\{r\alpha,2\}})$ and $\mathscr{O}(h + N^{-\min\{r\alpha,2\}})$, respectively, where, h, N, α, r is the total number of spatial parameter, the time-fractional order coefficient, and the time grid constant. Finally, three numerical examples are provided to illustrate our theoretical results.

AMS subject classifications: 65M10, 78A48

Key words: Time-fractional, interface problems, finite element, $L2$-1_σ method, weak singularity.

1. Introduction

Since fractional order derivatives have spatial heterogeneity and non-local nature, a large number of researchers apply them to complex models in various fields, including economics, chemistry, and biology [3, 11, 16]. Some models with fractional order derivatives can be found in interface problems. For example, transport in fractured reservoir [28], melting and solidification processes [31], and anomalous diffusion models of drug release [27].

In this paper, we deal with the following model problem. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. Assume that Ω contains two subdomains $\Omega_i \subset \Omega, i = 1, 2$ divided by a smooth interface Γ, cf. Fig. 1.

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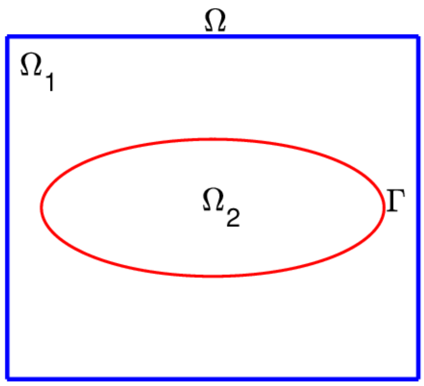

Figure 1: The domain and the interface.

Then, we consider the following initial boundary value problem for the time-fractional diffusion problems:

$$\begin{aligned} &\partial_t^\alpha u - \nabla\cdot(\beta\nabla u) = f(\boldsymbol{x},t) && \text{in } \Omega_1\cup\Omega_2, \quad T \geqslant t > 0,\\ &u(\boldsymbol{x},t) = 0 && \text{on } \partial\Omega, \quad T \geqslant t > 0,\\ &u(\boldsymbol{x},0) = u_0 && \text{in } \Omega\times\{0\} \end{aligned} \tag{1.1}$$

with the jump conditions

$$\begin{aligned} &[u]|_\Gamma := u_1|_\Gamma - u_2|_\Gamma = 0,\\ &[\beta\nabla u\cdot\boldsymbol{n}]|_\Gamma := \beta_1\nabla u_1\cdot\boldsymbol{n}|_\Gamma - \beta_2\nabla u_2\cdot\boldsymbol{n}|_\Gamma = g, \end{aligned} \tag{1.2}$$

where $\alpha \in (0,1)$ is time-fractional order coefficient, $\boldsymbol{n}$ is the unit normal vector to the interface Γ, u_1 and u_2 are the restriction of u on Ω_1 and Ω_2, respectively. Assume that the coefficient function $\beta = \beta(\boldsymbol{x}) : \Omega \to \mathbb{R}^{2\times 2}$ is symmetric and piecewise constant on each subdomain, i.e.

$$\beta(\boldsymbol{x}) = \begin{cases} \beta_1 & \text{for} \quad \boldsymbol{x}\in\Omega_1,\\ \beta_2 & \text{for} \quad \boldsymbol{x}\in\Omega_2, \end{cases}$$

and $\beta \in L^\infty(\Omega)$ satisfies

$$m|\xi|^2 \leqslant \xi^T\beta(\boldsymbol{x})\xi \leqslant M|\xi|^2 \quad \text{for all} \quad \xi\in\mathbb{R}^2, \quad \boldsymbol{x}\in\Omega \tag{1.3}$$

with constants $m, M > 0$.

In (1.1), the Caputo fractional derivative $\partial_t^\alpha u(\boldsymbol{x},t)$ is defined by

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u(\boldsymbol{x},s)}{\partial s}ds = \int_0^t \omega_{1-\alpha}(t-s)\cdot\frac{\partial u(\boldsymbol{x},s)}{\partial s}ds, \tag{1.4}$$

where $\omega_\mu(t) := t^{\mu-1}/\Gamma(\mu)$ and Γ is the Gamma function. Note that

$$\int_0^t \omega_\mu(s)ds = \omega_{\mu+1}(t), \quad \omega_\mu'(t) = \omega_{\mu-1}(t), \quad t > 0.$$