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Faster Deterministic Pseudoinverse-Free Block Extension of Motzkin Method for Large Consistent Linear Systems

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Abstract. Recently, a fast deterministic block Kaczmarz (FDBK) method which uses a greedy criterion of the row selections and contains pseudoinverse-free computation is presented. In this work, we introduce a maximum residual rule into FDBK and develop a new block Kaczmarz method which is also considered as a fast deterministic pseudoinverse-free block extension of Motzkin (FBEM) method. In addition, we prove that FBEM converges linearly to the unique least-norm solution of the linear systems. Furthermore, by incorporating the Polyak momentum technique into the FBEM iteration method, we establish an accelerated variant of FBEM (mFBEM) and show its global linear convergence. Numerical examples using artificial and real datasets demonstrate the effectiveness of FBEM as well as mFBEM.

AMS subject classifications: 65F10

Key words: Block Kaczmarz method, consistent linear system, maximum residual, heavy ball momentum.

1. Introduction

In this paper, we construct new block Kaczmarz-type methods for solving the large-scale consistent linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$ is a real matrix without zero rows, $b \in \mathbb{R}^m$ a real m-dimensional vector, and $x \in \mathbb{R}^n$ an unknown vector.

Recently, with block iterative methods excessive success in modeling many problems of big size, where data is frequently distributed over a network and only partially accessible at

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a time, it attracted attention of many researchers. The block Kaczmarz (BK) method [12] is a natural block variant of basic Kaczmarz method [14]. At each iteration, we choose a row submatrix with index subset $\mathscr{J}_k$ of the coefficient matrix A, project the current iteration point x_k onto the solution space $\{x \in \mathbb{R}^n \mid A_{\mathscr{J}_k} x = b_{\mathscr{J}_k}\}$, and compute the next iterate x_{k+1} as

$$x_{k+1} = x_k + A^{\dagger}_{\mathscr{J}_k}(b_{\mathscr{J}_k} - A_{\mathscr{J}_k} x_k),$$

where $A^{\dagger}_{\mathscr{J}_k}$ denotes the Moore-Penrose pseudoinverse of the submatrix $A_{\mathscr{J}_k}$.

Strohmer and Vershynin [30] proposed a randomized Kaczmarz (RK) method, which converges with linear rate in expectation. The RK method has been successfully applied to many practical problems and reignited many researchers to extend similar techniques to other iterative methods. Various acceleration strategies are established in [2–4, 6, 9], and the convergence analysis was extended to inconsistent, underdetermined or rank-deficient linear systems [13, 18, 22] (see also [1, 5] for other randomized iterative methods). In [23], the randomized block Kaczmarz (RBK) method was presented for solving linear least-squares problems, which was considered as a synthesis of the Elfving's block-iterative method and randomization. In [32], Wu presented the two-subspace REK method (TREK) for solving large-scale linear least-squares problems, which does not require any row or column paving. Recently, Niu and Zheng [25] developed a greed block Kaczmarz (GBK) method based on an approximate maximum distance (MD) greedy rule [26]. Adopting greedy probability criterion and almost-maximal residual control, Liu and Gu [16] proposed some greedy randomized block Kaczmarz methods for consistent linear systems (1.1). Zhang and Li [33] generalized the sampling Kaczmarz Motzkin (SKM) method and established the block sampling Kaczmarz Motzkin (BSKM) method for consistent linear systems. We note that GBK and BSKM do not need to pre-determine a partition of the row indices of the coefficient matrix A. For inconsistent linear systems, Needell *et al.* [24] presented a randomized double BK (RDBK) method. Recently, Rebrova and Needell [28] established a block Gaussian version based on block Gaussian sketch. Numerical results demonstrate the convergence rate can be significantly improved if appropriate blocks of the coefficient matrix A are provided. The main drawback of projection-based block methods is that computing the pseudoinverse of the submatrix is costly and difficult to parallelize at each iteration.

A randomized average block Kaczmarz (RaBK) method established by Necoara [21], is a pseudoinverse-free method where the $(k+1)$-th iteration x_{k+1} is computed as

$$x_{k+1} = x_k + \alpha_k \left(\sum_{i \in \tau_k} \omega_i^k \frac{b^{(i)} - A^{(i)} x_k}{\|A^{(i)}\|_2^2} (A^{(i)})^T \right), \quad k \ge 0 \tag{1.2}$$

with weights $\omega_i^k \in (0,1]$, $\sum_{i \in \tau_k} \omega_i^k = 1$ and the stepsize $\alpha_k > 0$. RaBK is very effective if a good sampling of the rows introduced into well-conditioned blocks. Moorman *et al.* [20] developed a randomized Kaczmarz method with averaging for inconsistent linear systems. Taking a convex combination of the RBK updates, Richtárik and Takáč [29] proposed a parallel RBK method. Shortly afterwards, Du *et al.* [10] proposed a simple