East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-004.200423

Vol. **14**, No. 1, pp. 79-103
February 2024

Numerical Methods for the Nonlinear Dirac Equation in the Massless Nonrelativistic Regime

Ying He[1,2], Yan Wang[3,*], Jerry Zhijian Yang[1] and Hongshuang Yin[1]

[1] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. China.*
[2] *Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing, 101408, P.R. China.*
[3] *School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, P.R. China.*

Received 4 January 2023; Accepted (in revised version) 20 April 2023.

Abstract. Numerical methods for the nonlinear Dirac equation (NDE) in the massless nonrelativistic regime are considered. In this regime, the equation contains a small dimensionless parameter $0 < \varepsilon \leq 1$, and its solution is highly oscillatory in time. We present and analyze traditional numerical schemes for the NDE, including finite difference methods, time-splitting methods and exponential integrators. Error analysis indicates that all these methods require an ε-dependent time-step size to achieve an optimal convergence order. Utilizing an operator splitting technique, we propose a uniformly accurate (UA) scheme. The scheme enables first-order convergence in time for all $\varepsilon \in (0,1]$ without restrictions on time-step size. Error estimates for the UA scheme are rigorously established and numerical results confirm the properties of the method.

AMS subject classifications: 35Q41, 65M12, 65M70

Key words: Nonlinear Dirac equation, uniformly accurate, finite difference method, time-splitting method, exponential integrator.

1. Introduction

The equation derived by Paul Dirac [21, 22] for describing spin-1/2 massive particles was named after him. It plays an important role in particle physics and relativistic quantum mechanics since then. It predicts the existence of positrons, and it is consistent with both the principle of quantum mechanics and the theory of special relativity. Later on, in 1938, Ivanenko [30] introduced a nonlinear Dirac equation by taking into account the self-interaction of particles. It has received considerable attention in mathematical and

*Corresponding author. *Email addresses:* heymath@whu.edu.cn (Y. He), wang.yan@ccnu.edu.cn (Y. Wang), zjyang.math@whu.edu.cn (Z. Yang), 2019202010048@whu.edu.cn (H. Yin)

physical studies [25, 31], especially on solitary wave solutions [2, 28, 39, 46]. Recently, the Dirac equation and the NDE attract renewed interests since they can be adopted to study graphene and Bose-Einstein condensates [1, 23, 26].

In this work, we consider the following one-dimensional NDE [5,7,10,21,22] on a torus $\mathbb{T} = \mathbb{R}/(2\pi)$ with periodic boundary conditions:

$$\begin{aligned} i\hbar\partial_t u(t,x) &= -ic\hbar\alpha\partial_x u(t,x) + mc^2\beta u(t,x) \\ &\quad + eVu(t,x) + F\big(u(t,x)\big), \quad t>0, \quad x\in\mathbb{T}, \\ u(0,x) &= u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{T}, \end{aligned}$$

where

$$u := u(t,x) = \big(u_1(t,x), u_2(t,x)\big)^T : [0,+\infty)\times\mathbb{T}\to\mathbb{C}^2$$

is the complex-valued vector wave function of the spinor field, $V := V(x)$ the real-valued electrical potential, $\hbar$ the Planck constant, c the speed of light, m the mass, and e the unit charge. We take the nonlinearity as $F(u) = \lambda(u^*\beta u)\beta u$ with $\lambda\in\mathbb{R}$ denoting the strength of the nonlinear interaction [43] and $u^* = \overline{u}^T$, while $\overline{u}$ denotes the complex conjugate of u, and α, β are the Pauli matrices

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the nondimensionalization

$$\tilde{x} = \frac{x}{x_s}, \quad \tilde{t} = \frac{t}{t_s}, \quad \tilde{V} = \frac{V}{A_s}, \quad \tilde{u} = \frac{u}{u_s},$$

where x_s, $t_s = m_s x_s^2/\hbar$, $A_s = m_s x_s^2/et_s^2$, $u_s = x_s^{-1/2}$ and m_s are respectively dimensionless length, time, potential, spinor field, and mass units — cf. [4, 5], and removing tilde $\sim$ everywhere, we arrive at a dimensionless form of the nonlinear Dirac equation — viz.

$$\begin{aligned} i\partial_t u(t,x) &= -i\frac{1}{\varepsilon}\alpha\partial_x u(t,x) + \delta\beta u(t,x) \\ &\quad + Vu(t,x) + F\big(u(t,x)\big), \quad t>0, \quad x\in\mathbb{T}, \\ u(0,x) &= u_0(x), \qquad\qquad\qquad\qquad\qquad x\in\mathbb{T} \end{aligned} \tag{1.1}$$

with $\delta = m_0/\varepsilon^2$. Note that $0<\varepsilon, m_0\le 1$ the dimensionless parameters defined by

$$\varepsilon := \frac{x_s}{t_s c} = \frac{v_s}{c}, \quad m_0 := \frac{m}{m_s},$$

where $v_s = x_s/t_s$ is the dimensionless velocity unit, ε the ratio between the wave velocity and the speed of light — i.e. it is inversely proportional to the speed of light, and m_0 the ratio between the mass of the particle m and the dimensionless mass unit m_s.

Under different scaling, the Eq. (1.1) corresponds to different parameter regimes, including the standard (classical) regime ($\varepsilon = m_0 = 1$), the nonrelativistic regime ($m_0 =$